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# NON-EXISTENCE OF SOLUTIONS FOR THE PERIODIC CUBIC NLS BELOW $L^2$

ZIHUA GUO AND TADAHIRO OH

ABSTRACT. We prove non-existence of solutions for the cubic nonlinear Schrödinger equation (NLS) on the circle if initial data belong to  $H^s(\mathbb{T}) \setminus L^2(\mathbb{T})$  for some  $s \in (-\frac{1}{8}, 0)$ . The proof is based on establishing an a priori bound on solutions to a renormalized cubic NLS in negative Sobolev spaces via the short time Fourier restriction norm method.

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## 1. INTRODUCTION

**1.1. The cubic nonlinear Schrödinger equation on  $\mathbb{T}$  and  $\mathbb{R}$ .** We consider the Cauchy problem for the cubic nonlinear Schrödinger equation (NLS) on the circle  $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ :

$$\begin{cases} i\partial_t u - \partial_x^2 u \pm |u|^2 u = 0 \\ u|_{t=0} = u_0, \end{cases} \quad (x, t) \in \mathbb{T} \times \mathbb{R}, \quad (1.1)$$

where  $u$  is a complex-valued function. The Cauchy problem (1.1) has been studied extensively from both theoretical and applied points of view. It is known to be one of the simplest partial differential equations (PDEs) with complete integrability [1, 2, 19]. In the following, however, we only discuss analytical aspects of (1.1) without using the complete integrable structure of the equation.

It is well known that (1.1) enjoys several symmetries. In the following, we concentrate on the scaling symmetry and the Galilean symmetry. The scaling symmetry states that if  $u(x, t)$  is a solution to (1.1) on  $\mathbb{R}$  with initial condition  $u_0$ , then  $u^\lambda(x, t) = \lambda^{-1}u(\lambda^{-1}x, \lambda^{-2}t)$

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is also a solution to (1.1) with the  $\lambda$ -scaled initial condition  $u_0^\lambda(x) = \lambda^{-1}u_0(\lambda^{-1}x)$ . Associated to the scaling symmetry, there is a scaling-critical Sobolev index  $s_c$  such that the homogeneous  $\dot{H}^{s_c}$ -norm is invariant under the dilation symmetry. It is commonly conjectured that a PDE is ill-posed in  $H^s$  for  $s < s_c$ . In the case of the one-dimensional cubic NLS, the scaling-critical Sobolev index is  $s_c = -\frac{1}{2}$  and Christ-Colliander-Tao [12] exhibited a norm inflation phenomenon in  $H^s(\mathbb{R})$  for  $s < -\frac{1}{2}$ .<sup>1</sup> While there is no dilation symmetry in the periodic setting, the heuristics provided by the scaling argument plays an important role even in the periodic setting. For example, a norm inflation result for (1.1) on  $\mathbb{T}$  analogous to [12] also holds for  $s < s_c = -\frac{1}{2}$ .

The Galilean symmetry states that if  $u(x, t)$  is a solution to (1.1) on  $\mathbb{R}$  with initial condition  $u_0$ , then  $u^\beta(x, t) = e^{i\frac{\beta}{2}x}e^{i\frac{\beta^2}{4}t}u(x + \beta t, t)$  is also a solution to (1.1) with the modulated initial condition  $u_0^\beta(x) = e^{i\frac{\beta}{2}x}u_0(x)$ . This symmetry also holds on the circle for  $\beta \in 2\mathbb{Z}$ . Note that the  $L^2$ -norm is invariant under the Galilean symmetry.<sup>2</sup> This induces another critical regularity  $s_{\text{crit}}^\infty = 0$ . Indeed, there is a strong dichotomy in the behavior of solutions for  $s \geq 0$  and  $s < 0$ .

Bourgain [3] introduced the so-called Fourier restriction norm method via the  $X^{s,b}$ -spaces (see (3.1) below) and proved local well-posedness of (1.1) in  $L^2(\mathbb{T})$ . Thanks to the  $L^2$ -conservation, this result was immediately extended to global well-posedness in  $L^2(\mathbb{T})$ . On the other hand, (1.1) is known to be ill-posed below  $L^2(\mathbb{T})$ . Burq-Gérard-Tzvetkov [5] showed that the solution map  $\Phi_t : u_0 \in H^s \mapsto u(t) \in H^s$  is not locally uniformly continuous if  $s < 0$ . See also [10]. Moreover, Christ-Colliander-Tao [11] and Molinet [31] proved that the solution map is indeed discontinuous if  $s < 0$ . Our main result in this paper states an even stronger form of ill-posedness holds true for (1.1) in negative Sobolev spaces.

**Theorem 1.1** (Non-existence of solutions for the cubic NLS). *Let  $s \in (-\frac{1}{8}, 0)$  and  $u_0 \in H^s(\mathbb{T}) \setminus L^2(\mathbb{T})$ . Then, for any  $T > 0$ , there exists no distributional solution  $u \in C([-T, T]; H^s(\mathbb{T}))$  to the cubic NLS (1.1) such that*

- (i)  $u|_{t=0} = u_0$ ,
- (ii) *There exist smooth global solutions  $\{u_n\}_{n \in \mathbb{N}}$  to (1.1) such that  $u_n \rightarrow u$  in  $C([-T, T]; H^s(\mathbb{T}))$  as  $n \rightarrow \infty$ .*

Note that the condition (ii) is a natural condition to impose since it would follow from the continuity of the solution map:  $u_0 \in H^s(\mathbb{T}) \mapsto u \in C([-T, T]; H^s(\mathbb{T}))$ , which is one of the essential components in the well-posedness theory of evolution equations. On the one hand, the previous ill-posedness results [5, 10, 11, 12, 31] treated smooth functions, at least in  $L^2(\mathbb{T})$ , to construct examples. On the other hand, the proof of Theorem 1.1 is based on establishing a priori bound on solutions to a renormalized cubic NLS (see (1.3) below) in negative Sobolev spaces. See Theorem 1.2 in Subsection 1.2.

In making sense of (1.1) on  $\mathbb{T} \times [-T, T]$  in the distributional sense, we need to assume that a solution  $u(t)$  is a priori in  $L^3(\mathbb{T})$  for almost every  $t \in [-T, T]$ . Thus, the usual construction

<sup>1</sup>This norm inflation also holds at the scaling critical regularity  $s = -\frac{1}{2}$  on both  $\mathbb{R}$  and  $\mathbb{T}$ . See Kishimoto [27] and Oh-Wang [34]. See also a recent result by Carles-Kappeler [7] for a norm inflation in the Fourier-Lebesgue space  $\mathcal{FL}^{s,p}(\mathbb{T})$  for  $s < -\frac{2}{3}$  and  $p \in [1, \infty]$ . These norm inflation results also hold for the Wick ordered cubic NLS defined in (1.3).

<sup>2</sup>In fact, any Fourier-Lebesgue  $\mathcal{FL}^{0,p}$ -norm defined in (1.4) is invariant under the Galilean symmetry.

of solutions in low regularities employs an auxiliary space-time function space  $X_T$ .<sup>3</sup> For example, the construction in [3] assumes a priori that solutions are in  $L^4_{x,t}(\mathbb{T} \times [-T, T]) \subset X^{0, \frac{1}{2}+}([-T, T])$ . Theorem 1.1 asserts non-existence of solutions below  $L^2(\mathbb{T})$  *even if* we assume a priori that  $u$  belongs to some auxiliary function space  $X_T$ .

Recently, there has been a significant development in probabilistic construction of local-in-time and global-in-time solutions to dispersive and hyperbolic PDEs. See, for example, [6, 17]. Such probabilistic arguments often allowed us to go below certain regularity thresholds such as a scaling critical regularity, below which the equations are known to be ill-posed deterministically. We would like to point out that the ill-posedness result stated in Theorem 1.1 is much stronger than ordinary ill-posedness results such as the failure of (local uniform) continuity of a solution map. In particular, Theorem 1.1 states that it is *not* possible to construct solutions to the cubic NLS (1.1) for initial data below  $L^2(\mathbb{T})$  *even probabilistically*.

Lastly, let us compare the situation with the non-periodic case. Tsutsumi [36] proved global well-posedness of (1.1) in  $L^2(\mathbb{R})$ . There are several results [26, 10] showing that the cubic NLS on  $\mathbb{R}$  is ‘mildly ill-posed’ below  $L^2$  in the sense that the solution map is not locally uniformly continuous if  $s < 0$ . There is, however, no ill-posedness result below  $L^2$  for the cubic NLS on  $\mathbb{R}$ , contradicting either existence, uniqueness, or continuous dependence. For example, while Molinet’s ill-posedness result [31] for the cubic NLS on  $\mathbb{T}$  is based on showing discontinuity of the solution map on  $L^2(\mathbb{T})$  endowed with weak topology, Goubet-Molinet [18] showed that the solution map for the cubic NLS on  $\mathbb{R}$  is weakly continuous on  $L^2(\mathbb{R})$ . Moreover, Christ-Colliander-Tao [13] and Koch-Tataru [28, 29] proved existence (without uniqueness) of solutions to the cubic NLS on  $\mathbb{R}$  in negative Sobolev spaces. Note that solutions constructed in [13, 28, 29] satisfy the conditions (i) and (ii) in Theorem 1.1, showing a sharp contrast between the periodic and non-periodic cases.

**1.2. The Wick ordered cubic NLS on  $\mathbb{T}$ .** Given a global solution  $u \in C(\mathbb{R}; L^2(\mathbb{T}))$  to (1.1), we define the following invertible gauge transformation:

$$u(t) \mapsto \mathcal{G}(u)(t) := e^{\mp 2it\mu(u)} u(t), \quad (1.2)$$

where  $\mu(u)$  is defined by

$$\mu(u) = \oint |u(x, t)|^2 dx := \frac{1}{2\pi} \int_{\mathbb{T}} |u(x, t)|^2 dx.$$

Thanks to the  $L^2$ -conservation,  $\mu(u)$  is defined, independently of  $t \in \mathbb{R}$ , as long as  $u_0 \in L^2(\mathbb{T})$ . A direct computation shows that the gauged function, which we still denote by  $u$ , solves the following *Wick ordered cubic NLS*:

$$\begin{cases} i\partial_t u - \partial_x^2 u \pm (|u|^2 - 2 \oint |u|^2 dx) u = 0 \\ u|_{t=0} = u_0, \end{cases} \quad (x, t) \in \mathbb{T} \times \mathbb{R}. \quad (1.3)$$

Conversely, given a global solution  $u \in C(\mathbb{R}; L^2(\mathbb{T}))$  to (1.3), we see that  $\mathcal{G}^{-1}(u)$  solves the original cubic NLS (1.1). Such a gauge transformation, however, does not make sense below  $L^2(\mathbb{T})$  and thus we cannot freely convert solutions of (1.3) into solutions of (1.1). In other words, the Wick ordered cubic NLS (1.3) arises from the standard cubic NLS (1.1) by choosing another gauge on the phase space such that they are equivalent (only) in

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<sup>3</sup> There is a more relaxed notion of weak solutions in the extended sense [8, 9]. In [23], we constructed weak solutions in the extended sense in  $L^2(\mathbb{T})$  without any auxiliary function space. Moreover, the result in [23] yields unconditional uniqueness in  $H^s(\mathbb{T})$  for  $s \geq \frac{1}{6}$ .

$L^2(\mathbb{T})$ . Note that this renormalization of the nonlinearity in (1.3) is canonical, appearing in Euclidean quantum field theory, and (1.3) first appeared in the work of Bourgain [4] for studying the invariant Gibbs measure for the defocusing cubic NLS on  $\mathbb{T}^2$ .

This specific choice of gauge for (1.3) removes a certain singular component from the cubic nonlinearity. As a result, the Wick ordered cubic NLS (1.3) is known to behave better than the cubic NLS (1.1) outside  $L^2(\mathbb{T})$ . In fact, the standard cubic NLS on  $\mathbb{R}$  and the Wick ordered cubic NLS (1.3) on  $\mathbb{T}$  share many common features. For example, just like the cubic NLS on  $\mathbb{R}$  [26, 10, 18], the solution map for the Wick ordered cubic NLS (1.3) on  $\mathbb{T}$  is known to be weakly continuous in  $L^2$  [32], while it is ‘mildly ill-posed’ in the sense that the solution map is not locally uniformly continuous below  $L^2$  [17]. See [32] for more discussion on this issue. In particular, we proposed in [32] that the Wick ordered cubic NLS (1.3) is the right model to study outside  $L^2(\mathbb{T})$ . As with any renormalization procedure or gauge choice, we stress that this is a matter of choice, since (1.1) and (1.3) are not equivalent outside  $L^2$ . The examples in [32] and Theorem 1.1 above provide supporting evidences for our choice.

Our main goal below is to establish an a priori estimate on solutions to (1.3) and prove existence of solutions to (1.3) in negative Sobolev spaces.

**Theorem 1.2.** *Let  $s \in (-\frac{1}{8}, 0)$ . Given  $u_0 \in H^s(\mathbb{T})$ , there exist  $T = T(\|u_0\|_{H^s}) > 0$  and a solution  $u \in C([-T, T]; H^s(\mathbb{T}))$  to the Wick ordered cubic NLS (1.3).*

Previously, Christ [9] and Grünrock-Herr [20] proved local well-posedness of (1.3) in  $\mathcal{FL}^{s,p}(\mathbb{T})$  for  $s = 0$  and  $p < \infty$ , where the Fourier-Lebesgue space  $\mathcal{FL}^p(\mathbb{T})$  is defined by the norm

$$\|f\|_{\mathcal{FL}^{s,p}(\mathbb{T})} = \|\langle n \rangle^s \widehat{f}(n)\|_{\ell^p(\mathbb{Z})}. \quad (1.4)$$

Note that  $\mathcal{FL}^{0,p}(\mathbb{T}) \supsetneq L^2(\mathbb{T})$  for  $p > 2$ . In the context of negative Sobolev spaces  $H^s(\mathbb{T})$ ,  $s < 0$ , Colliander-Oh [17] proved almost sure local well-posedness for  $s > -\frac{1}{3}$  and almost sure global well-posedness for  $s > -\frac{1}{12}$  with respect to the canonical Gaussian measures on  $H^s(\mathbb{T})$ . In Theorem 1.2, we claim only existence of solutions to (1.3) in negative Sobolev spaces, i.e. the uniqueness issue remains open. This is analogous to the situation for the standard cubic NLS on  $\mathbb{R}$  [13, 28, 29]. The proof of Theorem 1.2 occupies most of the remaining part of this paper: Sections 2-8. Once we prove Theorem 1.2, the proof of Theorem 1.1 immediately follows from the a priori bound on solutions to (1.3) in negative Sobolev spaces and inverting the gauge transformation (1.2). See Section 9.

The proof of Theorem 1.2 is based on establishing an a priori bound on (smooth) solutions to (1.3) in negative Sobolev spaces via the short time Fourier restriction norm method. Then, we use a compactness argument to construct solutions to (1.3) (without uniqueness). Here, the short time Fourier restriction norm method simply means that we use dyadically defined  $X^{s,b}$ -type spaces with suitable localization in time, depending on the dyadic size of spatial frequencies. A precursor of this method appears in the work of Koch-Tzvetkov [30], where localization in time was combined with the Strichartz norms. This method has been very effective in establishing a priori bounds on solutions in low regularities (yielding even uniqueness in some cases), in particular, where a solution map is known to fail to be locally uniformly continuous. See [13, 24, 28, 21, 29, 22, 25].

Let us briefly discuss the difference between the usual Fourier restriction norm method and the short time Fourier restriction norm method. Consider the following model evolution equation:  $u_t - Lu = \mathcal{N}(u)$ , where  $\mathcal{N}(u)$  is a homogenous nonlinearity of degree  $p$ . Then,

the usual Fourier restriction norm method requires the following two estimates:

$$\begin{aligned} \text{Linear:} \quad & \|u\|_{X^s(T)} \lesssim \|u_0\|_{H^s} + \|\mathcal{N}(u)\|_{N^s(T)}, \\ \text{Nonlinear:} \quad & \|\mathcal{N}(u)\|_{N^s(T)} \lesssim \|u\|_{X^s(T)}^p, \end{aligned}$$

where  $X^s(T) = X^{s,b}([-T, T])$  and  $N^s(T) = X^{s,b-1}([-T, T])$ ,  $b > \frac{1}{2}$ , in the simplest setting. Then, a fixed point argument yields well-posedness in  $H^s$ .<sup>4</sup> With the short time Fourier restriction norm method, one now needs to establish the following three estimates:

$$\text{Linear:} \quad \|u\|_{F^s(T)} \lesssim \|u\|_{E^s(T)} + \|\mathcal{N}(u)\|_{N^s(T)}, \quad (1.5)$$

$$\text{Nonlinear:} \quad \|\mathcal{N}(u)\|_{N^s(T)} \lesssim \|u\|_{F^s(T)}^p, \quad (1.6)$$

$$\text{Energy:} \quad \|u\|_{E^s(T)}^2 \lesssim \|u_0\|_{H^s}^2 + \|u\|_{F^s(T)}^{p+1}, \quad (1.7)$$

where  $E^s(T) \approx L^\infty([-T, T]; H^s(\mathbb{T}))$ . Then, these three estimates yield an a priori bound on solutions in  $H^s$ , which allows us to prove existence of solutions by a compactness argument. In order to improve the regularity  $s$ , one may apply a normal form reduction and establish a better energy estimate at the expense of introducing higher order nonlinear terms. This is analogous to considering higher ordered modified energies in the  $I$ -method developed by Colliander-Keel-Staffilani-Takaoka-Tao [15, 16]. See Section 7. In the following, we use the formulation introduced by Ionescu-Kenig-Tataru [24]. In Section 3, we provide the precise definitions of the function spaces  $F^s(T)$ ,  $N^s(T)$ , and  $E^s(T)$ .

**Remark 1.3.** In order to establish uniqueness of the solutions constructed in Theorem 1.2, one needs to establish an energy estimate  $\|u - v\|_{E^s(T)}$  on the difference of two solutions  $u$  and  $v$ . On the one hand, the conservation of the  $L^2$ -norm hints an almost conservation of the energy  $\|u\|_{E^s(T)}$  for  $s < 0$ , analogously to the  $I$ -method. On the other hand, the  $L^2$ -norm of the difference of two solutions  $u$  and  $v$  is not conserved under (1.3). This provides an essential difficulty in establishing an energy estimate on the difference of two solutions for  $s < 0$ .

**Remark 1.4.** For  $u_0 \notin L^2(\mathbb{T})$ , the renormalized nonlinearity in (1.3) can be formally written as

$$(|u|^2 - 2 \cdot \infty)u = 0.$$

What is important in this renormalization procedure is to *subtract off the right amount of infinity*. For example, a gauge transformation:  $\mathcal{G}(u)(t) := e^{\mp i\gamma t \mu(u)} u(t)$  with a parameter  $\gamma \in \mathbb{R}$  gives rise to the following renormalized cubic NLS:

$$i\partial_t u - \partial_x^2 u \pm (|u|^2 - \gamma \int |u|^2 dx)u = 0. \quad (1.8)$$

For  $u_0 \in L^2(\mathbb{T})$ , (1.8) is equivalent to (1.1) and (1.3). If  $u_0 \notin L^2(\mathbb{T})$ , however, they are not equivalent. In particular, if  $u_0 \in H^s(\mathbb{T}) \setminus L^2(\mathbb{T})$  for some  $s \in (-\frac{1}{8}, 0)$ , then a slight modification of the proof of Theorem 1.1 yields non-existence of solutions for (1.8) unless  $\gamma = 2$ . This shows that “ $2 \cdot \infty$ ” is the right amount to subtract in the renormalization procedure. See also [32] for the quantum field theoretic derivation of (1.3).

**Remark 1.5.** One can easily modify the argument in this paper and show that Theorem 1.1 also holds for the following fourth order dispersive analogue of the cubic NLS:

$$i\partial_t u + \partial_x^4 u \pm |u|^2 u = 0, \quad (x, t) \in \mathbb{T} \times \mathbb{R}. \quad (1.9)$$

---

<sup>4</sup>Strictly speaking, one needs to gain a smallness factor  $T^\theta$ ,  $\theta > 0$ , or assume small data to close the argument. For simplicity, however, we present only the essential part of the estimates.

On the one hand, by establishing a relevant  $L^4$ -Strichartz estimate and combining with the  $L^2$ -conservation, one can show that (1.9) is globally well-posed in  $L^2(\mathbb{T})$ . On the other hand, the example in [5, 10] basically yields the failure of local uniform continuity of the solution map in  $H^s(\mathbb{T})$  to (1.9) if  $s < 0$ . By considering the Wick ordered version of the fourth order cubic NLS:

$$i\partial_t u + \partial_x^4 u \pm (|u|^2 - 2 \int |u|^2 dx)u = 0, \quad (1.10)$$

one can show that Theorem 1.2 also holds for (1.10) (even for lower values of  $s < 0$ ). This essentially follows from the following algebraic observation: under  $n = n_1 - n_2 + n_3$ , we have

$$\begin{aligned} \Phi_4(\bar{n}) &:= -n^4 + n_1^4 - n_2^4 + n_3^4 \\ &= -(n - n_1)(n - n_3)(n_1^2 + n_2^2 + n_3^2 + n^2 + 2(n_1 + n_3)^2). \end{aligned} \quad (1.11)$$

Compare this with the phase function  $\Phi(\bar{n})$  for the cubic NLS defined in (2.5) below. In particular, the only resonant contribution, i.e.  $\Phi_4(\bar{n}) = 0$ , comes from  $n = n_1$  or  $n = n_3$ , just like the cubic NLS. Moreover, we have  $|\Phi_4(\bar{n})| \gtrsim |\Phi(\bar{n})|$  for any  $(n_1, n_2, n_3, n) \in \mathbb{Z}^4$  with  $n = n_1 - n_2 + n_3$ . As a corollary to Theorem 1.2 for (1.10), we see that Theorem 1.1 also holds for the fourth order cubic NLS (1.9). See [33, 35] for more discussions on (1.9) and (1.10). Lastly, consider the higher order dispersive analogue of (1.1):

$$i\partial_t u + (i\partial_x)^k u \pm |u|^2 u = 0.$$

When  $k$  is an odd integer, there is (at least) another resonant contribution coming from  $n = -n_2$  and thus Theorems 1.1 and 1.2 are not directly applicable in this case. When  $k$  is an even integer, it may be of interest to investigate if the phase function  $\Phi_k(\bar{n}) := (-1)^{\frac{k}{2}}(-n^k + n_1^k - n_2^k + n_3^k)$  admits a factorization similar to (1.11) and (2.5) such that results analogous to Theorems 1.1 and 1.2 may hold. We, however, do not pursue this issue further here.

This paper is organized as follows. In Sections 2 and 3, we introduce notations and the function spaces along with their basic properties. We prove the linear estimate (1.5) in Section 4 and the key trilinear estimate (1.6), with  $p = 3$ , in Sections 5 and 6. The energy estimate (1.7) with a higher order correction term is established in Section 7. In Section 8, we combine the estimates (1.5), (1.6), and (1.7) and prove Theorem 1.2. Finally, we use the a priori estimate on solutions from Theorem 1.2 and prove Theorem 1.1 in Section 9.

Some of the estimates presented in this paper, in particular those in Sections 3 and 4, are by now standard. For the sake of completeness of the paper, we present the proofs in details. Since our argument is of local-in-time nature, there is no difference between the focusing and defocusing cases. Hence, we assume that (1.3) is defocusing (with the  $+$  sign in (1.1) and (1.3)) in the following.

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## 2. NOTATIONS

For  $a, b > 0$ , we use  $a \lesssim b$  to mean that there exists  $C > 0$  such that  $a \leq Cb$ . By  $a \sim b$ , we mean that  $a \lesssim b$  and  $b \lesssim a$ . We also use  $a+$  (and  $a-$ ) to denote  $a + \varepsilon$  (and  $a - \varepsilon$ ), respectively, for arbitrarily small  $\varepsilon \ll 1$ .

Given  $u \in \mathcal{S}'$ , we use  $\widehat{u}$  and  $\mathcal{F}(u)$  to denote the space-time Fourier transform of  $u$  given by

$$\widehat{u}(n, \tau) = \int_{\mathbb{T} \times \mathbb{R}} e^{-inx} e^{-it\tau} u(x, t) dx dt.$$

Moreover, we use  $\mathcal{F}_x$  and  $\mathcal{F}_t$  to denote the Fourier transforms with respect to the spatial and temporal variables, respectively. When there is no confusion, we may simply use  $\widehat{u}$  or  $\mathcal{F}(u)$  to denote the spatial, temporal, or space-time Fourier transform of  $u$ , depending on the context. In dealing with the spatial Fourier transform, we often denote  $\widehat{u}(n, t)$  by  $\widehat{u}_n(t)$ .

For  $k \in \mathbb{Z}_+ := \mathbb{Z} \cap [0, \infty)$ , let  $I_0 = \{\xi : |\xi| < 1\}$  and  $I_k = \{\xi : 2^{k-1} \leq |\xi| < 2^k\}$  if  $k \geq 1$ . For  $k \in \mathbb{Z}_+$  and  $j \geq 0$ , let

$$D_{k,j} = \{(n, \tau) \in \mathbb{Z} \times \mathbb{R} : n \in I_k, \tau - n^2 \in I_j\}$$

and  $D_{k,\leq j} = \bigcup_{j' \leq j} D_{k,j'}$ . Lastly, we define  $D_{\leq j}$  by

$$D_{\leq j} = \bigcup_{k=0}^{\infty} D_{k,\leq j}.$$

We use  $\mathbf{P}_k$  to denote the projection operator on  $L^2(\mathbb{T})$  defined by  $\widehat{\mathbf{P}_k u}(n) = \mathbf{1}_{I_k}(n) \widehat{u}(n)$ . With a slight abuse of notation, we also use  $\mathbf{P}_k$  to denote the projection operator on  $L^2(\mathbb{T} \times \mathbb{R})$  given by  $\mathcal{F}(\mathbf{P}_k u)(n, \tau) = \mathbf{1}_{I_k}(n) \mathcal{F}(u)(n, \tau)$ . For  $\ell \in \mathbb{Z}$ , let

$$\mathbf{P}_{\leq \ell} = \sum_{0 \leq k \leq \ell} \mathbf{P}_k \quad \text{and} \quad \mathbf{P}_{\geq \ell} = \sum_{k \geq \ell} \mathbf{P}_k.$$

Let  $\eta_0 : \mathbb{R} \rightarrow [0, 1]$  be an even smooth cutoff function supported on  $[-\frac{8}{5}, \frac{8}{5}]$  such that  $\eta_0 \equiv 1$  on  $[-\frac{5}{4}, \frac{5}{4}]$ . We define  $\eta$  by  $\eta(\xi) = \eta_0(\xi) - \eta_0(2\xi)$ , and set  $\eta_k(\xi) = \eta(2^{-k}\xi)$  for  $k \in \mathbb{Z}$ . Namely,  $\eta_k$  is supported on  $\{\frac{5}{4} \cdot 2^{k-1} \leq |\xi| \leq \frac{8}{5} \cdot 2^k\}$ . As before, we define  $\eta_{\leq \ell} = \sum_{k \leq \ell} \eta_k$ , etc.

Given a set of indices such as  $j_i$  and  $k_i$ ,  $i = 1, \dots, 4$ , we use  $j_i^*$  and  $k_i^*$  to denote the decreasing rearrangements of these indices. Also, given a set of frequencies  $n_i$ ,  $i = 1, \dots, 4$ , we use  $n_i^*$  to denote the decreasing rearrangements of  $|n_i|$ ,  $i = 1, \dots, 4$ .

In the following, we use  $S(t) = e^{-it\partial_x^2}$  to denote the solution operator to the linear Schrödinger equation:  $i\partial_t u - \partial_x^2 u = 0$ . Namely, for  $\phi \in L^2(\mathbb{T})$ , we have

$$S(t)\phi = \sum_{n \in \mathbb{Z}} e^{inx + in^2 t} \widehat{\phi}(n).$$

Lastly, let us discuss the renormalized nonlinearity in (1.3). The nonlinearity on the right-hand side of (1.3) can be written as

$$\begin{aligned} \mathfrak{N}(u) &= \mathfrak{N}(u, u, u) := (|u|^2 - 2 \oint |u|^2 dx) u \\ &=: \mathcal{N}(u, u, u) - \mathcal{R}(u, u, u), \end{aligned}$$



where the non-resonant part  $\mathcal{N}$  and the resonant part  $\mathcal{R}$  are defined by

$$\mathcal{N}(u_1, u_2, u_3)(x, t) = \sum_{\substack{n_2 \neq n_1, n_3}} \widehat{u}_1(n_1, t) \overline{\widehat{u}_2(n_2, t)} \widehat{u}_3(n_3, t) e^{i(n_1 - n_2 + n_3)x}, \quad (2.1)$$

$$\mathcal{R}(u_1, u_2, u_3)(x, t) = \sum_n \widehat{u}_1(n, t) \overline{\widehat{u}_2(n, t)} \widehat{u}_3(n, t) e^{inx}. \quad (2.2)$$

Here, the condition  $n_2 \neq n_1, n_3$  in the sum for  $\mathcal{N}(u)$  is a shorthand notation for  $n_2 \neq n_1$  and  $n_2 \neq n_3$ .

In establishing an energy estimate in Section 7, we use the following interaction representation  $a$  (of  $u$ ) on  $\mathbb{T} \times \mathbb{R}$ :

$$a(t) := S(-t)u(t) = e^{it\partial_x^2}u(t). \quad (2.3)$$

On the Fourier side, we have  $\widehat{a}_n(t) = e^{-itn^2}\widehat{u}_n(t)$ ,  $n \in \mathbb{Z}$ . For simplicity of notations, we use  $a_n(t)$  to denote  $\widehat{a}_n(t)$  in the following. The use of the interaction representation allows us to illustrate the connection between the Poincaré-Dulac normal form reduction discussed in [23] and the method of adding correction terms in the  $I$ -method [15, 16]. With this notation, (1.3) can be written as

$$\begin{aligned} \partial_t a_n &= i \sum_{\substack{n=n_1-n_2+n_3 \\ n_2 \neq n_1, n_3}} e^{-i\Phi(\bar{n})t} a_{n_1} \bar{a}_{n_2} a_{n_3} - i|a_n|^2 a_n \\ &=: i \mathbf{N}(a)(n, t) - i \mathbf{R}(a)(n, t), \end{aligned} \quad (2.4)$$

where the phase function  $\Phi(\bar{n})$  is defined by

$$\begin{aligned} \Phi(\bar{n}) &:= \Phi(n_1, n_2, n_3, n) = n^2 - n_1^2 + n_2^2 - n_3^2 \\ &= 2(n_2 - n_1)(n_2 - n_3) = 2(n - n_1)(n - n_3). \end{aligned} \quad (2.5)$$

Here, the last two equalities hold under  $n = n_1 - n_2 + n_3$ . Noting that

$$\mathbf{N}(a)(n, t) = e^{-in^2 t} \mathcal{F}(\mathcal{N}(u))(n, t) \quad \text{and} \quad \mathbf{R}(a)(n, t) = e^{-in^2 t} \mathcal{F}(\mathcal{R}(u))(n, t),$$

it follows from (2.5) that  $\mathcal{N}$  defined in (2.1) indeed corresponds to the non-resonant part (i.e.  $\Phi(\bar{n}) \neq 0$ ) of the nonlinearity  $\mathfrak{N}(u)$  and  $\mathcal{R}$  defined in (2.2) corresponds to the resonant part.

### 3. FUNCTION SPACES AND THEIR BASIC PROPERTIES

In [3], Bourgain introduced the weighted space-time Sobolev spaces called the  $X^{s,b}$ -spaces via the norm:

$$\|u\|_{X^{s,b}(\mathbb{T} \times \mathbb{R})} = \|\langle n \rangle^s \langle \tau - n^2 \rangle^b \widehat{u}(n, \tau)\|_{\ell_n^2 L_\tau^2(\mathbb{Z} \times \mathbb{R})}. \quad (3.1)$$

In terms of the interaction representation defined in (2.3), we simply have  $\|u\|_{X^{s,b}(\mathbb{T} \times \mathbb{R})} = \|\langle \partial_x \rangle^s \langle \partial_t \rangle^b a\|_{L^2(\mathbb{T} \times \mathbb{R})}$ . The  $X^{s,b}$ -spaces and their variants have been very effective in studying nonlinear evolution equations in low regularity settings. In the following, we define the  $X^{s,b}$ -spaces adapted to short time scales. These spaces were first introduced by Ionescu-Kenig-Tataru [24] in the context of the KP-I equation. Also, see Christ-Colliander-Tao [13] for similar definitions.

For  $k \in \mathbb{Z}_+$ , we define the dyadic  $X^{s,b}$ -type spaces  $X_k(\mathbb{Z} \times \mathbb{R})$ :

$$\begin{aligned} X_k &= \left\{ f_k \in L^2(\mathbb{Z} \times \mathbb{R}) : f_k(n, \tau) \text{ is supported on } I_k \times \mathbb{R}, \right. \\ &\quad \left. \text{and } \|f_k\|_{X_k} := \sum_{j=0}^{\infty} 2^{\frac{j}{2}} \|\eta_j(\tau - n^2) f_k(n, \tau)\|_{\ell_n^2 L_\tau^2} < \infty \right\}. \end{aligned} \quad (3.2)$$

We list some properties of the space  $X_k$ . It follows easily from the definition that if  $f_k \in X_k$  for some  $k \in \mathbb{Z}_+$ , then

$$\left\| \int_{\mathbb{R}} |f_k(n, \tau)| d\tau \right\|_{\ell_n^2} \lesssim \|f_k\|_{X_k}. \quad (3.3)$$

Letting  $g_k(n, \tau) = f_k(n, \tau + n^2)$ , we have

$$\int_{\mathbb{R}} \|g_k(n, \tau)\|_{\ell_n^2} d\tau \lesssim \|f_k\|_{X_k}. \quad (3.4)$$

Moreover, for  $k, \ell \in \mathbb{Z}_+$  and  $f_k \in X_k$ , we have then

$$\begin{aligned} &\sum_{j=\ell+1}^{\infty} 2^{\frac{j}{2}} \left\| \eta_j(\tau - n^2) \int_{\mathbb{R}} |f_k(n, \tau')| 2^{-\ell} (1 + 2^{-\ell} |\tau - \tau'|)^{-4} d\tau' \right\|_{\ell_n^2 L_\tau^2} \\ &\quad + 2^{\frac{\ell}{2}} \left\| \eta_{\leq \ell}(\tau - n^2) \int_{\mathbb{R}} |f_k(n, \tau')| 2^{-\ell} (1 + 2^{-\ell} |\tau - \tau'|)^{-4} d\tau' \right\|_{\ell_n^2 L_\tau^2} \lesssim \|f_k\|_{X_k}, \end{aligned} \quad (3.5)$$

where the implicit constant is independent of  $k$  and  $\ell$ . See [21] for the proof of (3.5). In particular, for  $k, \ell \in \mathbb{Z}_+$ ,  $t_0 \in \mathbb{R}$ ,  $f_k \in X_k$  and  $\gamma \in \mathcal{S}(\mathbb{R})$ , we have

$$\|\mathcal{F}[\gamma(2^\ell(t - t_0)) \cdot \mathcal{F}^{-1}(f_k)]\|_{X_k} \lesssim \|f_k\|_{X_k}. \quad (3.6)$$

Note that the implicit constant in (3.6) is also independent of  $k, \ell$ , and  $t_0$ .

At spatial frequencies  $|n| \sim 2^k$ , we will use the  $X^{s,b}$ -structure given by the  $X_k$ -norm, localized on the time scale  $\sim 2^{-[\alpha k]}$ , where  $\alpha > 0$  is to be determined later. Here,  $[x]$  denotes the integer part of  $x$ . For  $k \in \mathbb{Z}_+$  we define the spaces  $F_k^\alpha$  and  $N_k^\alpha$  by

$$\begin{aligned} F_k^\alpha &= \left\{ u \in L^2(\mathbb{T} \times \mathbb{R}) : \widehat{u}(n, \tau) \text{ is supported in } I_k \times \mathbb{R}, \right. \\ &\quad \left. \text{and } \|u\|_{F_k^\alpha} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F}[\eta_0(2^{[\alpha k]}(t - t_k)) \cdot u]\|_{X_k} < \infty \right\}, \\ N_k^\alpha &= \left\{ u \in L^2(\mathbb{T} \times \mathbb{R}) : \widehat{u}(n, \tau) \text{ is supported in } I_k \times \mathbb{R}, \right. \\ &\quad \left. \text{and } \|u\|_{N_k^\alpha} = \sup_{t_k \in \mathbb{R}} \|(\tau - n^2 + i2^{[\alpha k]})^{-1} \mathcal{F}[\eta_0(2^{[\alpha k]}(t - t_k)) \cdot u]\|_{X_k} < \infty \right\}. \end{aligned}$$

Next, we define local-in-time versions of these spaces in the usual way. For  $T \in (0, 1]$ , we define the local-in-time spaces  $F_k^\alpha(T)$  and  $N_k^\alpha(T)$  by

$$\begin{aligned} F_k^\alpha(T) &= \left\{ u \in C([-T, T]; L^2(\mathbb{T})) : \|u\|_{F_k^\alpha(T)} = \inf_{\tilde{u}=u \text{ on } \mathbb{T} \times [-T, T]} \|\tilde{u}\|_{F_k^\alpha} \right\}, \\ N_k^\alpha(T) &= \left\{ u \in C([-T, T]; L^2(\mathbb{T})) : \|u\|_{N_k^\alpha(T)} = \inf_{\tilde{u}=u \text{ on } \mathbb{T} \times [-T, T]} \|\tilde{u}\|_{N_k^\alpha} \right\}. \end{aligned}$$

Here, the infimum is taken over all extensions  $\tilde{u} \in C_0(\mathbb{R}; L^2(\mathbb{T}))$ . So far, we have defined the dyadic function spaces. We now assemble these dyadic spaces in a straight forward

manner using the Littlewood-Paley decomposition. For  $s \in \mathbb{R}$  and  $T \in (0, 1]$ , we define the spaces  $F^{s,\alpha}(T)$  and  $N^{s,\alpha}(T)$  by

$$F^{s,\alpha}(T) = \left\{ u : \|u\|_{F^{s,\alpha}(T)}^2 = \sum_{k=0}^{\infty} 2^{2sk} \|\mathbf{P}_k u\|_{F_k^\alpha(T)}^2 < \infty \right\},$$

$$N^{s,\alpha}(T) = \left\{ u : \|u\|_{N^{s,\alpha}(T)}^2 = \sum_{k=0}^{\infty} 2^{2sk} \|\mathbf{P}_k u\|_{N_k^\alpha(T)}^2 < \infty \right\}.$$

In order to deal with these short-time spaces  $F^{s,\alpha}(T)$  and  $N^{s,\alpha}(T)$ , we need to define the corresponding energy space. For  $s \in \mathbb{R}$  and  $u \in C([-T, T]; H^s(\mathbb{T}))$ , let

$$\|u\|_{E^s(T)}^2 = \|\mathbf{P}_0 u(0)\|_{L^2(\mathbb{T})}^2 + \sum_{k \geq 1} \sup_{t_k \in [-T, T]} 2^{2sk} \|\mathbf{P}_k u(t_k)\|_{L^2(\mathbb{T})}^2.$$

Note that the energy space  $E^s(T)$  is independent of the parameter  $\alpha > 0$ . This space is essentially the usual energy space  $C([-T, T]; H^s(\mathbb{T}))$  but with a logarithmic difference.

We conclude this section by stating basic embeddings involving the  $F^{s,\alpha}$ -spaces. It follows immediately from (3.6) that if  $\alpha_1 \geq \alpha_2$ , then we have  $F^{s,\alpha_2}(T) \subset F^{s,\alpha_1}(T)$ .

The following lemma shows that a smooth time cutoff supported on a interval of size  $\sim 2^{-[\alpha k]}$  acts boundedly on  $N_k^\alpha$ .

**Lemma 3.1.** *Let  $\alpha \geq 0$ ,  $k \in \mathbb{Z}_+$ ,  $t_k \in \mathbb{R}$ , and  $\gamma \in \mathcal{S}(\mathbb{R})$ . Then, we have*

$$\|(\tau - n^2 + i2^{[\alpha k]})^{-1} \mathcal{F}[\gamma(2^{[\alpha k]}(t - t_k)) \cdot \mathcal{F}^{-1}(f_k)]\|_{X_k} \lesssim \|(\tau - n^2 + i2^{[\alpha k]})^{-1} f_k\|_{X_k} \quad (3.7)$$

for  $f_k$  supported on  $I_k \times \mathbb{R}$ . Here, the implicit constant is independent of  $\alpha, k$ , and  $t_k$ .

*Proof.* First, note that

$$|\tau - n^2 + i2^{[\alpha k]}|^{-1} (1 + 2^{-[\alpha k]} |\tau - \tau'|)^{-1} \lesssim |\tau' - n^2 + i2^{[\alpha k]}|^{-1}. \quad (3.8)$$

Then, (3.7) follows from (3.5) and (3.8).  $\square$

The next lemma shows that the  $F_k^\alpha$ -norm controls the  $L_t^\infty L_x^2$ -norm of a dyadic piece.

**Lemma 3.2.** *Let  $u$  be a function on  $\mathbb{T} \times \mathbb{R}$  such that  $\text{supp } \widehat{u} \subset I_k \times \mathbb{R}$ . Then, we have*

$$\|u\|_{L_t^\infty L_x^2} \lesssim \|u\|_{F_k^\alpha} \quad (3.9)$$

for any  $\alpha > 0$ . Similarly, we have

$$\|\mathcal{F}^{-1}[\eta_{\leq j}(\tau - n^2) \widehat{u}]\|_{L_t^\infty L_x^2} \lesssim \|u\|_{F_k^\alpha} \quad (3.10)$$

for any  $j \in \mathbb{Z}_+$ . Here, (3.10) also holds when we replace  $\eta_{\leq j}$  by  $\eta_j$  or  $\eta_{\geq j}$ .

*Proof.* Let  $t \in \mathbb{R}$ . Then, by (3.3), we have

$$\begin{aligned} \|u(x, t)\|_{L_x^2} &= \sup_{t_k} \|\eta_0(2^{[\alpha k]}(t - t_k)) \cdot u(x, t)\|_{L_x^2} \\ &\lesssim \sup_{t_k} \left\| \int |\mathcal{F}[\eta_0(2^{[\alpha k]}(t - t_k)) \cdot u](n, \tau)| d\tau \right\|_{\ell_n^2} \leq \|u\|_{F_k^\alpha}. \end{aligned}$$

The second estimate (3.10) follows from (3.9) once we note that

$$\|\mathcal{F}^{-1}[\eta_{\leq j}(\tau - n^2) \widehat{u}]\|_{L_t^\infty L_x^2} \lesssim \|u\|_{L_t^\infty L_x^2},$$

which follows from  $\mathcal{F}^{-1}[\eta_{\leq j}(\tau - n^2)](t) = 2^j \check{\eta}_0(2^j t) e^{itn^2}$  and Young's inequality.  $\square$

As a corollary to Lemma 3.2, we have the following control of the  $C([-T, T]; H^s)$ -norm by the  $F^{s, \alpha}(T)$ -norm.

**Lemma 3.3.** *Let  $s \in \mathbb{R}$ ,  $T \in (0, 1]$ , and  $\alpha > 0$ . Then, we have*

$$\sup_{t \in [-T, T]} \|u(t)\|_{H^s} \lesssim \|u\|_{F^{s, \alpha}(T)}. \quad (3.11)$$

*Proof.* For  $k \in \mathbb{Z}_+$ , let  $\tilde{u}_k$  be an extension of  $\mathbf{P}_k u$ . Then, by Lemma 3.2, we have

$$\|\mathbf{P}_k u(t)\|_{L_x^2} = \|\tilde{u}_k(t)\|_{L_x^2} \lesssim \|\tilde{u}_k\|_{F_k^\alpha} \quad (3.12)$$

for  $t \in [-T, T]$ . Then, (3.11) follows from (3.12) by taking an infimum over extensions  $\tilde{u}_k$  of  $\mathbf{P}_k(u)$ , summing over dyadic blocks, and taking a supremum in  $t \in [-T, T]$ .  $\square$

So far, we defined the function spaces with the modulation regularity  $\frac{1}{2}$ . In the following, we define the corresponding function spaces with the modulation regularity  $b$ . For  $k \in \mathbb{Z}_+$  and  $b \in \mathbb{R}$ , let  $X_k^b$  denote the dyadic  $X^{s, b}$ -type space analogous to  $X_k$ , whose norm is given by

$$\|f_k\|_{X_k^b} := \sum_{j=0}^{\infty} 2^{jb} \|\eta_j(\tau - n^2) f_k(n, \tau)\|_{\ell_n^2 L_\tau^2}$$

for  $f_k$  supported on  $I_k \times \mathbb{R}$ . By definition, we have  $X_k = X_k^{\frac{1}{2}}$ . Then, we define the spaces  $F_k^{b, \alpha}$  and  $F^{s, b, \alpha}(T)$  with  $X_k^b$ , just as we defined  $F_k^\alpha$  and  $F^{s, \alpha}(T)$  with  $X_k$ .

The following lemma shows that we obtain a small power of time localization at a slight expense of the regularity in modulation.

**Lemma 3.4.** *Let  $\alpha, T > 0$  and  $b < \frac{1}{2}$ . Then, we have*

$$\|\mathbf{P}_k u\|_{F_k^{b, \alpha}} \lesssim T^{\frac{1}{2}-b-} \|\mathbf{P}_k u\|_{F_k^\alpha}$$

for any function  $u$  supported on  $\mathbb{T} \times [-T, T]$ .

*Proof.* Let  $\chi(t)$  be the characteristic function on  $[-1, 1]$  and  $\chi_T(t) = \chi(T^{-1}t)$ , i.e.  $\chi_T$  is the characteristic function of  $[-T, T]$ . Note that

$$\|\widehat{\chi}_T\|_{L^q} \sim T^{1-\frac{1}{q}} \quad (3.13)$$

for  $q > 1$ . For fixed  $t_k \in \mathbb{R}$ , let  $v_k = \eta_0(2^{[\alpha k]}(t - t_k)) \cdot \mathbf{P}_k(u)$ . Then, we have  $v_k = \chi_T \cdot v_k$  and it suffices to show

$$\|\mathcal{F}[\chi_T \cdot v]\|_{X_k^b} \lesssim T^{\frac{1}{2}-b-} \|\mathcal{F}(v_k)\|_{X_k}. \quad (3.14)$$

In the following, we simply use  $v$  to denote  $v_k$ . Write  $v = \sum_{j' \in \mathbb{Z}_+} v_{j'}$ , where  $v_{j'} = \mathcal{F}^{-1}[\eta_{j'}(\tau - n^2) \widehat{v}]$ . For any  $\varepsilon > 0$ , we have

$$\|\mathcal{F}[\chi_T \cdot v]\|_{X_k^b} \lesssim \sup_{j \in \mathbb{Z}_+} 2^{(b+\varepsilon)j} \|\eta_j(\tau - n^2) \mathcal{F}[\chi_T \cdot v]\|_{\ell_n^2 L_\tau^2}. \quad (3.15)$$

Fix  $j \in \mathbb{Z}_+$ . First, we consider the contribution from  $j' \geq j - 5$ . By Hölder's inequality (with  $\theta = \frac{1}{2} - b - \varepsilon > 0$ ) and Young's inequality with (3.13), we have

$$\begin{aligned} (3.15) &\lesssim \sup_{j \in \mathbb{Z}_+} 2^{\frac{j}{2}} \sum_{j'} \|\eta_j(\tau - n^2) \mathcal{F}[\chi_T \cdot v_{j'}]\|_{\ell_n^2 L_\tau^{\frac{2}{1-2\theta}}} \lesssim \sup_{j \in \mathbb{Z}_+} \sum_{j' \geq j-5} 2^{\frac{j'}{2}} \|\mathcal{F}[\chi_T \cdot v_{j'}]\|_{\ell_n^2 L_\tau^{\frac{2}{1-2\theta}}} \\ &\lesssim T^\theta \sup_{j \in \mathbb{Z}_+} \sum_{j' \geq j-5} 2^{\frac{j'}{2}} \|\widehat{v}_{j'}\|_{\ell_n^2 L_\tau^2} \leq T^{\frac{1}{2}-b-\varepsilon} \|\mathcal{F}(v)\|_{X_k}. \end{aligned}$$

Next, we consider the contribution from  $j' < j - 5$ . With  $\mathcal{F}(\chi_T \cdot v)(n, \tau) = \int \widehat{\chi}_T(\tau - \tau') \widehat{v}(n, \tau') d\tau'$ , we have  $|\tau - \tau'| \sim 2^j$  in this case. Then, by Young's inequality with (3.13), we have

$$\begin{aligned} (3.15) &\lesssim \sup_{j \in \mathbb{Z}_+} \left\| \eta_j(\tau - n^2) \int_{|\tau - \tau'| \sim 2^j} |\tau - \tau'|^{b+\varepsilon} |\widehat{\chi}_T(\tau - \tau')| |\widehat{v}(n, \tau')| d\tau' \right\|_{\ell_n^2 L_\tau^2} \\ &\lesssim \sup_{j \in \mathbb{Z}_+} \left\| |\tau|^{b+\varepsilon} \widehat{\chi}_T \right\|_{L_\tau^2} \sum_{|\tau| \sim 2^j} \|\widehat{v}_{j'}\|_{\ell_n^2 L_\tau^1} \lesssim T^{\frac{1}{2}-b-\varepsilon} \|\mathcal{F}(v)\|_{X_k}. \end{aligned}$$

This proves (3.14).  $\square$

As a corollary to the proof of Lemma 3.4, we obtain the following lemma.

**Lemma 3.5.** *Let  $k \in \mathbb{Z}_+$ . Then, for any interval  $I = [t_1, t_2] \subset \mathbb{R}$ , we have*

$$\sup_{j \in \mathbb{Z}_+} 2^{\frac{j}{2}} \left\| \eta_j(\tau - n^2) \mathcal{F}[\mathbf{1}_I(t) \cdot u] \right\|_{\ell_n^2 L_\tau^2} \lesssim \|\mathcal{F}(u)\|_{X_k},$$

where the implicit constant is independent of  $k$  and  $I$ .

*Proof.* Set  $b + \varepsilon = \frac{1}{2}$  in (3.15).  $\square$

As in [24], for any  $k \in \mathbb{Z}_+$  we define the set  $S_k^\alpha$  of  $k$ -acceptable time multiplication factors:

$$S_k^\alpha = \{m_k : \mathbb{R} \rightarrow \mathbb{R} : \|m_k\|_{S_k^\alpha} = \sum_{j=0}^{10} 2^{-j[\alpha k]} \|\partial^j m_k\|_{L^\infty} < \infty\}. \quad (3.16)$$

**Lemma 3.6.** *Let  $\alpha > 0$  and  $m_k$  be a  $k$ -acceptable time multiplication factor. Then, we have*

$$\|m_k(t)u_k\|_{F_k^\alpha} \lesssim \|m_k\|_{S_k^\alpha} \|u_k\|_{F_k^\alpha}, \quad (3.17)$$

$$\|m_k(t)u_k\|_{N_k^\alpha} \lesssim \|m_k\|_{S_k^\alpha} \|u_k\|_{N_k^\alpha} \quad (3.18)$$

for any function  $u_k$  on  $\mathbb{T} \times \mathbb{R}$  such that  $\text{supp } \widehat{u}_k \subset I_k \times \mathbb{R}$ .

*Proof.* Fix  $t_k \in \mathbb{R}$ . Let  $\gamma : \mathbb{R} \rightarrow [0, 1]$  be a smooth cutoff function supported on  $[-2, 2]$  such that  $\gamma \equiv 1$  on  $[-\frac{8}{5}, \frac{8}{5}]$ . Then, we have  $\eta_0(2^{[\alpha k]}(t - t_k)) \cdot m_k u_k = \gamma(2^{[\alpha k]}(t - t_k)) \cdot m_k v_k$ , where  $v_k(t) = \eta_0(2^{[\alpha k]}(t - t_k)) \cdot u_k(t)$ . Integrating by parts several times, we have

$$|\mathcal{F}[\gamma(2^{[\alpha k]}(\cdot - t_k)) \cdot m_k](\tau)| \lesssim 2^{-[\alpha k]} (1 + 2^{-[\alpha k]} |\tau|)^{-4} \|m_k\|_{S_k^\alpha}.$$

Then, (3.17) follows from (3.5). The second estimate (3.18) follows from a similar argument with (3.8). This completes the proof of Lemma 3.6.  $\square$

The following lemma shows a kind of almost orthogonality property of  $X_k$ . Due to the  $\ell^1$ -based Besov nature of the temporal regularity in  $X_k$ , we have the second terms on the right-hand sides in (3.19) and (3.20). See Lemma 6.4 in [13] for a related result.

**Lemma 3.7** (Almost orthogonality). *Let  $\gamma : \mathbb{R} \rightarrow [0, 1]$  be a smooth function supported on  $[-1, 1]$  such that*

$$\sum_{m \in \mathbb{Z}} \gamma(t - m) \equiv 1$$

for all  $t \in \mathbb{R}$ .

(a) *There exists  $C > 0$  such that*

$$\begin{aligned} \|\mathcal{F}(u)\|_{X_k} &\leq C \left( \sum_{m \in \mathbb{Z}} \|\mathcal{F}[\gamma(\lambda t - m) \cdot u]\|_{X_k}^2 \right)^{\frac{1}{2}} \\ &\quad + C \sum_{j=0}^{\infty} \left( \sum_{m \in \mathbb{Z}} 2^j \|\eta_j(\tau - n^2) \mathcal{F}[\gamma(\lambda t - m) \cdot u](n, \tau)\|_{\ell_n^2 L_\tau^2}^2 \right)^{\frac{1}{2}} \end{aligned} \quad (3.19)$$

for all  $\lambda \geq 1$ .

(b) *There exists  $C > 0$  such that*

$$\begin{aligned} \|\eta_{\geq \ell}(\tau - n^2)(\tau - n^2 + iA)^{-1} \mathcal{F}(u)\|_{X_k} &\leq C \left( \sum_{m \in \mathbb{Z}} \|(\tau - n^2 + iA)^{-1} \mathcal{F}[\gamma(\lambda t - m) \cdot u]\|_{X_k}^2 \right)^{\frac{1}{2}} \\ &\quad + C \sum_{j=0}^{\infty} \left( \sum_{m \in \mathbb{Z}} 2^j \|\eta_j(\tau - n^2)(\tau - n^2 + iA)^{-1} \mathcal{F}[\gamma(\lambda t - m) \cdot u](n, \tau)\|_{\ell_n^2 L_\tau^2}^2 \right)^{\frac{1}{2}} \end{aligned} \quad (3.20)$$

for all  $A, \lambda \geq 1$ , where  $\ell$  is the greatest integer such that  $2^\ell \leq \lambda$ .

**Remark 3.8.** In view of Minkowski's integral inequality:  $\ell_j^1 \ell_m^2 \subset \ell_m^2 \ell_j^1$ , we see that the first terms on the right-hand sides of (3.19) and (3.20) are controlled by the second terms and thus are not needed. We, however, keep the first terms on the right-hand sides of (3.19) and (3.20) as we can estimate many cases simply by them. Namely, the second terms are the correction terms needed to handle a few special cases.

We present the proof of Lemma 3.7 at the end of this section. In the following, we first discuss corollaries to Lemma 3.7.

**Corollary 3.9.** *Let  $\gamma$  be as in Lemma 3.7. Then, for  $b < \frac{1}{2}$ , there exists  $C = C(b) > 0$  such that for all functions  $u$  with  $\text{supp } \hat{u} \subset I_k \times \mathbb{R}$ , we have*

$$\|\mathcal{F}(u)\|_{X_k^b} \leq C \left( \sum_{m \in \mathbb{Z}} \|\mathcal{F}[\gamma(\lambda t - m) \cdot u]\|_{X_k}^2 \right)^{\frac{1}{2}}$$

for all  $\lambda \geq 1$ .

The proof of Corollary 3.9 is immediate from (3.19) and Cauchy-Schwarz inequality. The next lemma shows a relation between the  $F^{s,\alpha}$ -spaces at different time scales. Once again, a slight loss of the regularity in modulation is necessary due to the  $\ell^1$ -based Besov nature of the  $X_k$ -norm.

**Lemma 3.10.** *Suppose that  $\alpha \geq \beta \geq 0$ ,  $s, r \in \mathbb{R}$  with  $r \leq \frac{\beta - \alpha}{2} + s$ . Then, for  $b < \frac{1}{2}$ , there exists  $C = C(b) > 0$  such that*

$$\|u\|_{F^{r,b,\beta}(T)} \leq C \|u\|_{F^{s,\alpha}(T)} \quad (3.21)$$

for all functions  $u$  on  $\mathbb{T} \times \mathbb{R}$  and  $T \in (0, 1]$ .

*Proof.* From the definition of  $F^{s,\alpha}$ , it suffices to prove

$$2^{kr} \|u\|_{F_k^{b,\beta}} \lesssim 2^{ks} \|u\|_{F_k^\alpha}$$

for  $k \in \mathbb{Z}_+$ . From the support conditions for  $\eta_0$  and  $\gamma$ , we have

$$\eta_0(2^{[\beta k]}(t - t_k)) \cdot u(t) = \sum_{|m| \leq C2^{k(\alpha-\beta)}} \gamma(2^{[\alpha k]}(t - t_k) - m) \cdot \eta_0(2^{[\beta k]}(t - t_k)) \cdot u(t)$$

for some  $C > 0$ . Then, by Corollary 3.9 and (3.6), we have

$$\begin{aligned} \|u\|_{F_k^{b,\beta}} &= \sup_{t_k \in \mathbb{R}} \|\mathcal{F}[\eta_0(2^{[\beta k]}(t - t_k)) \cdot u]\|_{X_k^b} \\ &\leq C(b) \sup_{t_k \in \mathbb{R}} \left( \sum_{|m| \leq C2^{k(\alpha-\beta)}} \|\mathcal{F}[\gamma(2^{[\alpha k]}(t - t_k) - m) \cdot \eta_0(2^{[\beta k]}(t - t_k)) \cdot u]\|_{X_k} \right)^{\frac{1}{2}} \\ &\lesssim 2^{\frac{k(\alpha-\beta)}{2}} \|u\|_{F_k^\alpha}. \end{aligned}$$

Hence, (3.21) follows as long as  $r \leq \frac{\beta-\alpha}{2} + s$ .  $\square$

We conclude this section by presenting the proof of Lemma 3.7.

*Proof of Lemma 3.7.* (a) Let  $u_m = \gamma(\lambda t - m) \cdot u$ . Then, we have

$$\|\mathcal{F}(u)\|_{X_k} = \sum_{j=0}^{\infty} 2^{\frac{j}{2}} \left\| \eta_j(\tau) \sum_{m \in \mathbb{Z}} \widehat{u}_m(n, \tau + n^2) \right\|_{\ell_n^2 L_\tau^2}. \quad (3.22)$$

Let  $v_m(n, \tau) = \widehat{u}_m(n, \tau + n^2)$ . Since  $\eta_0 \equiv 1$  on the support of  $\gamma$ , we have  $u_m = \eta_0(\lambda t - m) \cdot u_m$  and thus

$$v_m = \int v_m(n, s) e^{-i \frac{(\tau-s)m}{\lambda}} \lambda^{-1} \widehat{\eta}_0\left(\frac{\tau-s}{\lambda}\right) ds.$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left\| \eta_j(\tau) \sum_{m \in \mathbb{Z}} \widehat{u}_m(n, \tau + n^2) \right\|_{\ell_n^2 L_\tau^2}^2 \\ &= \sum_{m, m' \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int \eta_j^2(\tau) v_m(n, s) e^{-i \frac{(\tau-s)m}{\lambda}} \lambda^{-1} \widehat{\eta}_0\left(\frac{\tau-s}{\lambda}\right) \\ &\quad \times \overline{v_{m'}(n, s')} e^{i \frac{(\tau-s')m'}{\lambda}} \lambda^{-1} \overline{\widehat{\eta}_0}\left(\frac{\tau-s'}{\lambda}\right) ds ds' d\tau \\ &\lesssim \sum_{m, m' \in \mathbb{Z}} \int \|v_m(\cdot, s)\|_{\ell_n^2} \|v_{m'}(\cdot, s')\|_{\ell_n^2} |K_\lambda(s, s', m - m')| ds ds', \end{aligned} \quad (3.23)$$

where

$$K_\lambda(s, s', m - m') = \lambda^{-2} \int \eta_j^2(\tau) e^{-i \frac{\tau(m-m')}{\lambda}} \widehat{\eta}_0\left(\frac{\tau-s}{\lambda}\right) \overline{\widehat{\eta}_0}\left(\frac{\tau-s'}{\lambda}\right) d\tau \quad (3.24)$$

$$= \lambda^{-1} \int \eta_j^2(\lambda \zeta) e^{-i \zeta(m-m')} \widehat{\eta}_0(\zeta - \lambda^{-1}s) \overline{\widehat{\eta}_0}(\zeta - \lambda^{-1}s') d\zeta. \quad (3.25)$$

• **Case (i):** First, we consider the case  $\lambda \gtrsim 2^j$ .

Suppose that  $|m - m'| \lesssim \frac{\lambda}{2^j}$ . From (3.25), we have  $|K_\lambda| \lesssim \frac{2^j}{\lambda^2}$ . Then, using (3.4), we estimate (3.23) by

$$\sum_m \sum_{m': |m-m'| \lesssim \frac{\lambda}{2^j}} \frac{2^j}{\lambda^2} \left( \int \|v_m(\cdot, s)\|_{\ell_n^2} ds \right) \left( \int \|v_{m'}(\cdot, s')\|_{\ell_n^2} ds' \right) \lesssim \lambda^{-1} \sum_m \|\widehat{u}_m\|_{X_k}^2.$$

Hence, we obtain

$$(3.22) \lesssim \sum_{2^j \lesssim \lambda} 2^{\frac{j}{2}} \left( \lambda^{-1} \sum_m \|\widehat{u}_m\|_{X_k}^2 \right)^{\frac{1}{2}} \sim \left( \sum_{m \in \mathbb{Z}} \|\mathcal{F}[\gamma(\lambda t - m) \cdot u]\|_{X_k}^2 \right)^{\frac{1}{2}}.$$

When  $|m - m'| \gg \frac{\lambda}{2^j}$ , we integrate (3.25) by parts twice and obtain  $|K_\lambda| \lesssim 2^{-j} |m - m'|^{-2}$ . In this case, (3.23) is estimated by

$$2^{-j} \sum_m \sum_{m': |m - m'| \gg \frac{\lambda}{2^j}} \frac{1}{|m - m'|^2} \|\widehat{u}_m\|_{X_k} \|\widehat{u}_{m'}\|_{X_k} \lesssim \lambda^{-1} \sum_m \|\widehat{u}_m\|_{X_k}^2.$$

Hence, the same conclusion holds as before.

• **Case (ii):** Next, we consider the case  $\lambda \ll 2^j$ .

First, we consider the contribution from  $m = m'$ . In this case, the contribution to (3.22) is bounded by

$$\sum_{2^j \gg \lambda} 2^{\frac{j}{2}} \left( \sum_m \left\| \eta_j(\tau) \int \|v_m(n, s)\|_{\ell_n^2} \lambda^{-1} \widehat{\eta}_0\left(\frac{\tau - s}{\lambda}\right) ds \right\|_{L_\tau^2}^2 \right)^{\frac{1}{2}}. \quad (3.26)$$

When  $|s| \ll |\tau|$  or  $|s| \gg |\tau|$ , we have  $|\tau - s| \gtrsim 2^j$ . In this case, we have  $|\eta_0(\lambda^{-1}(\tau - s))| \lesssim \lambda^2 2^{-2j} \phi(\lambda^{-1}(\tau - s))$  for some  $\phi \in \mathcal{S}$ , and thus we have

$$(3.26) \lesssim \sum_{2^j \gg \lambda} 2^{\frac{j}{2}} \frac{\lambda}{2^{\frac{3}{2}j}} \left\{ \sum_m \left( \int \|v_m(n, s)\|_{\ell_n^2} ds \right)^2 \right\}^{\frac{1}{2}} \lesssim \left( \sum_m \|\widehat{u}_m\|_{X_k}^2 \right)^{\frac{1}{2}}.$$

When  $|s| \sim |\tau| \sim 2^j$ , it follows from Young's inequality that

$$\begin{aligned} (3.26) &\lesssim \sum_{2^j \gg \lambda} 2^{\frac{j}{2}} \left( \sum_m \|\eta_j(s) v_m(n, s)\|_{\ell_n^2 L_s^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \sum_j \left( \sum_m 2^j \|\eta_j(\tau - n^2) \widehat{u}_m(n, \tau)\|_{\ell_n^2 L_\tau^2}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Next, we consider the contribution from  $m \neq m'$ .

◦ Subcase (ii.1):  $|m - m'| \gg \frac{2^j}{\lambda}$ . In this case, integrating by parts three times, we have  $|K_\lambda| \lesssim \lambda^{-1} |m - m'|^{-3}$ . The contribution to (3.23) is estimated by

$$\sum_m \sum_{m': |m - m'| \gg \frac{2^j}{\lambda}} \frac{1}{\lambda |m - m'|^3} \left( \int \|v_m(n, s)\|_{\ell_n^2} ds \right) \left( \int \|v_{m'}(n, s')\|_{\ell_n^2} ds' \right) \lesssim \frac{\lambda}{2^{2j}} \sum_m \|\widehat{u}_m\|_{X_k}^2.$$

Hence, we obtain

$$(3.22) \lesssim \sum_{2^j \gg \lambda} 2^{\frac{j}{2}} \left( \frac{\lambda}{2^{2j}} \sum_m \|\widehat{u}_m\|_{X_k}^2 \right)^{\frac{1}{2}} \sim \left( \sum_{m \in \mathbb{Z}} \|\mathcal{F}[\gamma(\lambda t - m) \cdot u]\|_{X_k}^2 \right)^{\frac{1}{2}}.$$

◦ Subcase (ii.2):  $|m - m'| \lesssim \frac{2^j}{\lambda}$ .



When  $|s| \ll |\tau|$  or  $|s| \gg |\tau|$ , we have  $|\tau - s| \gtrsim 2^j$ . In this case, we have  $|K_\lambda| \lesssim \frac{\lambda^2}{2^{3j}}$  from (3.24). The contribution to (3.23) is estimated by

$$\sum_m \sum_{m': 0 < |m-m'| \lesssim \frac{2^j}{\lambda}} \frac{\lambda^2}{2^{3j}} \left( \int \|v_m(n, s)\|_{\ell_n^2} ds \right) \left( \int \|v_{m'}(n, s')\|_{\ell_n^2} ds' \right) \lesssim \frac{\lambda}{2^{2j}} \sum_m \|\widehat{u}_m\|_{X_k}^2.$$

Then, the result follows as in Subcase (ii.1). A similar argument holds when  $|s'| \ll |\tau|$  or  $|s'| \gg |\tau|$ .

Lastly, when  $|s| \sim |s'| \sim |\tau| \sim 2^j$ , we integrate (3.25) by parts four times and obtain

$$|K_\lambda| \lesssim |m - m'|^{-4} \lambda^{-1} (1 + \lambda^{-1} |s - s'|)^{-4}$$

By Young's inequality, the contribution to (3.23) is estimated by

$$\begin{aligned} & \sum_m \sum_{m': 0 < |m-m'| \lesssim \frac{2^j}{\lambda}} \frac{1}{|m - m'|^4} \|\eta_j(s) v_m(n, s)\|_{\ell_n^2 L_s^2} \\ & \quad \times \left\| \int \eta_j(s') \|v_{m'}(n, s')\|_{\ell_n^2} \lambda^{-1} (1 + \lambda^{-1} |s - s'|)^{-4} ds' \right\|_{L_s^2} \\ & \lesssim \sum_m \|\eta_j(s) v_m(n, \tau)\|_{\ell_n^2 L_\tau^2}^2. \end{aligned}$$

Hence, we obtain

$$(3.22) \lesssim \sum_j \left( \sum_m 2^j \|\eta_j(\tau - n^2) \widehat{u}_m(n, \tau)\|_{\ell_n^2 L_\tau^2}^2 \right)^{\frac{1}{2}}.$$

(b) Proceeding as in Part (a), we have

$$\|\eta_{\geq \ell}(\tau - n^2)(\tau - n^2 + iA)^{-1} \mathcal{F}(u)\|_{X_k} = \sum_{j=\ell}^{\infty} 2^{\frac{j}{2}} \left\| \eta_j(\tau) \sum_{m \in \mathbb{Z}} \frac{\widehat{u}_m(n, \tau + n^2)}{|\tau| + A} \right\|_{\ell_n^2 L_\tau^2}$$

and

$$\begin{aligned} & \left\| \eta_j(\tau) \sum_{m \in \mathbb{Z}} \frac{\widehat{u}_m(n, \tau + n^2)}{|\tau| + A} \right\|_{\ell_n^2 L_\tau^2}^2 \\ & \lesssim \sum_{m, m' \in \mathbb{Z}} \int \|v_m(\cdot, s)\|_{\ell_n^2} \|v_{m'}(\cdot, s')\|_{\ell_n^2} |\widetilde{K}_\lambda(s, s', m - m')| ds ds', \end{aligned}$$

where

$$\widetilde{K}_\lambda(s, s', m - m') = \lambda^{-2} \int \frac{\eta_j^2(\tau)}{(|\tau| + A)^2} e^{-i \frac{\tau(m-m')}{\lambda}} \widehat{\eta}_0\left(\frac{\tau - s}{\lambda}\right) \overline{\widehat{\eta}_0}\left(\frac{\tau - s'}{\lambda}\right) d\tau.$$

When  $|\tau| + A \gtrsim \max(|s|, |s'|)$ , we have

$$\frac{1}{(|\tau| + A)^2} \lesssim \frac{1}{|s| + A} \frac{1}{|s'| + A},$$

and thus we can proceed as in Part (a). Next, we consider the case  $|\tau| + A \ll \max(|s|, |s'|)$ . Without loss of generality, assume  $|s| \geq |s'|$ . In this case, we have  $|\tau - s| \sim |s| \sim |s| + A$ . Thus,

$$\left| \widehat{\eta}_0\left(\frac{\tau - s}{\lambda}\right) \right| \lesssim \frac{\lambda}{|s| + A} \phi\left(\frac{\tau - s}{\lambda}\right)$$

for some  $\phi \in \mathcal{S}$ . In particular, since we have  $\lambda \sim 2^\ell \leq 2^j$ , we have

$$\frac{1}{|\tau| + A} \frac{\lambda}{|s| + A} \lesssim \frac{1}{|s| + A}. \quad (3.27)$$

If  $|s'| \gg |\tau| + A$ , then we use (3.27) with  $|s'|$  in place of  $|s|$ . If  $|s'| \lesssim |\tau| + A$ , we use  $(|\tau| + A)^{-1} \lesssim (|s'| + A)^{-1}$ . Then, we can proceed as in Case (ii) of Part (a) in this case. This completes the proof of Lemma 3.7.  $\square$

#### 4. LINEAR ESTIMATE

In this section, we present a linear estimate associated to the Schrödinger equation. In particular, we estimate solutions to the nonhomogeneous linear Schrödinger equation.

**Proposition 4.1.** *Let  $T \in (0, 1]$ . Suppose that  $u \in C([-T, T]; H^\infty(\mathbb{T}))$  is a solution to the following nonhomogeneous linear Schrödinger equation:*

$$i\partial_t u - \partial_x^2 u = v \quad \text{on } \mathbb{T} \times (-T, T),$$

where  $v \in C([-T, T]; H^\infty(\mathbb{T}))$ . Then, for any  $s \in \mathbb{R}$  and  $\alpha \geq 0$ , we have

$$\|u\|_{F^{s, \alpha}(T)} \lesssim \|u\|_{E^s(T)} + \|v\|_{N^{s, \alpha}(T)}. \quad (4.1)$$

*Proof.* We follow the argument in [24, 21]. We first make some preliminary computations. Given  $\phi_k \in L^2(\mathbb{T})$  with  $\text{supp } \hat{\phi}_k \subset I_k$ , we have

$$\|\mathcal{F}[\eta_0(2^{[\alpha k]}(t - t_k)) \cdot S(t - t^*)\phi_k]\|_{X_k} \lesssim \|\phi_k\|_{L^2} \quad (4.2)$$

for any  $\alpha \geq 0$  and  $t_k, t^* \in \mathbb{R}$ . This easily follows from

$$\mathcal{F}[\eta_0(2^{[\alpha k]}(t - t_k)) \cdot S(t - t^*)\phi_k](n, \tau) = e^{-it_k(\tau - n^2)} 2^{-[\alpha k]} \hat{\eta}_0(2^{-[\alpha k]}(\tau - n^2)) e^{-it^* n^2} \hat{\phi}_k(n).$$

Moreover, a straightforward computation shows

$$\begin{aligned} & \mathcal{F}[\eta_0(2^{[\alpha k]}t) \int_0^t S(t - t') v_k(t') dt'](n, \tau) \\ &= C \int \hat{v}_k(n, \tau') \cdot 2^{-[\alpha k]} \frac{\hat{\eta}_0(2^{-[\alpha k]}(\tau - \tau')) - \hat{\eta}_0(2^{-[\alpha k]}(\tau - n^2))}{\tau' - n^2} d\tau' \end{aligned}$$

and

$$\begin{aligned} & \left| 2^{-[\alpha k]} \frac{\hat{\eta}_0(2^{-[\alpha k]}(\tau - \tau')) - \hat{\eta}_0(2^{-[\alpha k]}(\tau - n^2))}{\tau' - n^2} \cdot (\tau' - n^2 + i2^{[\alpha k]}) \right| \\ & \lesssim 2^{-[\alpha k]} (1 + 2^{-[\alpha k]} |\tau - \tau'|)^{-4} + 2^{-[\alpha k]} (1 + 2^{-[\alpha k]} |\tau - n^2|)^{-4}. \end{aligned}$$

Hence, from (3.3) and (3.5), we have

$$\left\| \mathcal{F}[\eta_0(2^{[\alpha k]}t) \int_0^t S(t - t') v_k(t') dt'] \right\|_{X_k} \lesssim \|(\tau - n^2 + i2^{[\alpha k]})^{-1} \cdot \mathcal{F}(v_k)\|_{X_k} \quad (4.3)$$

for  $v_k$  on  $\mathbb{T} \times \mathbb{R}$  with  $\text{supp } \hat{v}_k \subset I_k \times \mathbb{R}$ .

In view of the definitions, the square of the right-hand side of (4.1) is equivalent to

$$\|\mathbf{P}_0 u(0)\|_{L^2}^2 + \|\mathbf{P}_0 v\|_{N_0^\alpha(T)}^2 + \sum_{k \geq 1} \left( \sup_{t_k \in [-T, T]} 2^{2sk} \|\mathbf{P}_k u(t_k)\|_{L^2}^2 + 2^{2sk} \|\mathbf{P}_k v\|_{N_k^\alpha(T)}^2 \right).$$

Thus, it suffices to prove

$$\|\mathbf{P}_k u\|_{F_k^\alpha(T)} \lesssim \begin{cases} \|\mathbf{P}_0 u(0)\|_{L^2} + \|\mathbf{P}_0(v)\|_{N_0^\alpha(T)}, & k = 0, \\ \sup_{t_k \in [-T, T]} \|\mathbf{P}_k u(t_k)\|_{L^2} + \|\mathbf{P}_k v\|_{N_k^\alpha(T)}, & k \geq 1. \end{cases} \quad (4.4)$$

For  $k \in \mathbb{Z}_+$ , let  $\tilde{v}_k$  denote an extension of  $\mathbf{P}_k v$  such that

$$\|\tilde{v}_k\|_{N_k^\alpha} \leq C \|\mathbf{P}_k v\|_{N_k^\alpha(T)}. \quad (4.5)$$

By Lemma 3.6, we may assume that  $\tilde{v}_k$  is supported on  $\mathbb{T} \times [-T - 2^{-[\alpha k]-10}, T + 2^{-[\alpha k]-10}]$ . Indeed, let  $m(t)$  be a smooth cutoff function such that

$$m(t) = \begin{cases} 1, & \text{for } t \geq 1, \\ 0, & \text{for } t \leq 0. \end{cases}$$

Then, defining  $m_{k,-}$  and  $m_{k,+}$  by

$$\begin{aligned} m_{k,-}(t) &= m(2^{[\alpha k]+10}(t + T + 2^{-[\alpha k]-10})), \\ m_{k,+}(t) &= m(-2^{[\alpha k]+10}(t - T - 2^{-[\alpha k]-10})), \end{aligned}$$

we have  $m_{k,-}, m_{k,+} \in S_k^\alpha$ , where  $S_k^\alpha$  is defined in (3.16). Note that  $m_k(t) := m_{k,-}(t) \cdot m_{k,+}(t)$  is supported on  $[-T - 2^{-[\alpha k]-10}, T + 2^{-[\alpha k]-10}]$ , and is equal to 1 on  $[-T, T]$ . Since  $\tilde{v}_k$  is an extension of  $\mathbf{P}_k v$ ,  $m_k \cdot \tilde{v}_k$  is also an extension of  $\mathbf{P}_k v$ . Moreover, from (3.18), we have  $\|m_k \cdot \tilde{v}_k\|_{N_k^\alpha} \lesssim \|\tilde{v}_k\|_{N_k^\alpha} \leq C \|\mathbf{P}_k v\|_{N_k^\alpha(T)}$ . Hence, we assume that  $\tilde{v}_k$  is supported on  $\mathbb{T} \times [-T - 2^{-[\alpha k]-10}, T + 2^{-[\alpha k]-10}]$  in the following.

Next, we define an extension  $\tilde{u}_k$  of  $\mathbf{P}_k u$ . For  $k \geq 1$ , we define  $\tilde{u}_k$  by setting

$$\tilde{u}_k(t) = \begin{cases} \eta_0(2^{[\alpha k]+5}(t - T)) \left[ S(t - T) \mathbf{P}_k u(T) - i \int_T^t S(t - t') \tilde{v}_k(t') dt' \right], & t \geq T, \\ \eta_0(2^{[\alpha k]+5}(t + T)) \left[ S(t + T) \mathbf{P}_k u(-T) - i \int_{-T}^t S(t - t') \tilde{v}_k(t') dt' \right], & t \leq -T, \end{cases}$$

and  $\tilde{u}_k(t) = \mathbf{P}_k u(t)$  for  $t \in [-T, T]$ . When  $k = 0$ , define  $\tilde{u}_0$  by

$$\tilde{u}_0(t) = \begin{cases} \eta_0(2^5(t - T)) \left[ S(t) \mathbf{P}_0 u(0) - i \int_0^t S(t - t') \tilde{v}_0(t') dt' \right], & t \geq T, \\ \eta_0(2^5(t + T)) \left[ S(t) \mathbf{P}_0 u(0) - i \int_0^t S(t - t') \tilde{v}_0(t') dt' \right], & t \leq -T, \end{cases}$$

and  $\tilde{u}_0(t) = \mathbf{P}_0 u(t)$  for  $t \in [-T, T]$ .

Using (3.6), we can show that

$$\|\mathbf{P}_k u\|_{F_k^\alpha(T)} \lesssim \sup_{t_k \in [-T, T]} \|\mathcal{F}[\eta_0(2^{[\alpha k]}(t - t_k)) \cdot \tilde{u}_k]\|_{X_k}. \quad (4.6)$$

Namely, with this choice of  $\tilde{u}_k$ , we can restrict the domain of the supremum from  $t_k \in \mathbb{R}$  to  $t_k \in [-T, T]$ . In the following, we first prove (4.4), assuming (4.6). We then present the proof of (4.6) at the end.

First, we consider the case  $k = 0$ . Let  $\gamma : \mathbb{R} \rightarrow [0, 1]$  be a smooth cutoff function supported on  $[-1, 1]$  such that

$$\sum_{m \in \mathbb{Z}} \gamma(t - m) \equiv 1, \quad t \in \mathbb{R},$$

as in Lemma 3.7. Note that there exists  $C > 0$  such that

$$\sum_{|m| \leq C} \gamma(t - t_0 - m) = 1, \quad \text{on } \left[-\frac{8}{5} - T, \frac{8}{5} + T\right]$$

for  $t_0 \in [-T, T]$ . Then, by (4.6), (4.2), (4.3), and (4.5), we have

$$\begin{aligned} \|\mathbf{P}_0 u\|_{F_0^\alpha(T)} &\leq \sup_{t_0 \in [-T, T]} \left\| \mathcal{F} \left[ \eta_0(t - t_0) \cdot \left( S(t) \mathbf{P}_0 u(0) - i \int_0^t S(t - t') \tilde{v}_0(t') dt' \right) \right] \right\|_{X_0} \\ &\leq \|\mathbf{P}_0 u(0)\|_{L^2} \\ &\quad + \sup_{t_0 \in [-T, T]} \sum_{|m| \leq C} \left\| (\tau - n^2 + i)^{-1} \cdot \mathcal{F}[\gamma(t - t_0 - m) \cdot \tilde{v}_0] \right\|_{X_0} \\ &\lesssim \|\mathbf{P}_0 u(0)\|_{L^2} + \|\mathbf{P}_0 v\|_{N_0^\alpha(T)}, \end{aligned}$$

yielding (4.4). Note that we used  $\gamma(t - t_0 - m) = \gamma(t - t_0 - m) \eta_0(t - t_0 - m)$  and (3.6) in the last step.

Next, we consider the case  $k \geq 1$ . Given  $t_k \in [-T, T]$ , we write

$$\tilde{u}_k(t) = \eta_{T,k}(t) \left[ S(t - t_k) \mathbf{P}_k u(t_k) - i \int_{t_k}^t S(t - t') \tilde{v}_k(t') dt' \right],$$

where  $\eta_{T,k}(t)$  is defined by

$$\eta_{T,k}(t) = \eta_0(2^{[\alpha k] + 5}(t - T)) \mathbf{1}_{(T, \infty)} + \mathbf{1}_{[-T, T]} + \eta_0(2^{[\alpha k] + 5}(t + T)) \mathbf{1}_{(-\infty, -T)}.$$

Then, noting that

$$\eta_0(2^{[\alpha k]}(t - t_k)) = \eta_0(2^{[\alpha k]}(t - t_k)) \sum_{|m| \leq C} \gamma(2^{[\alpha k]}(t - t_k) - m),$$

we proceed with (4.2) and (4.3) and obtain

$$\begin{aligned} \|\mathbf{P}_k u\|_{F_k^\alpha(T)} &\lesssim \sup_{t_k \in [-T, T]} \|\mathbf{P}_k u(t_k)\|_{L^2} \\ &\quad + \sup_{t_k \in [-T, T]} \sum_{|m| \leq C} \left\| (\tau - n^2 + i 2^{[\alpha k]})^{-1} \cdot \mathcal{F}[\gamma(2^{[\alpha k]}(t - t_k) - m) \cdot \tilde{v}_0] \right\|_{X_0} \\ &\lesssim \sup_{t_k \in [-T, T]} \|\mathbf{P}_k u(t_k)\|_{L^2} + \|\mathbf{P}_k v\|_{N_k^\alpha(T)}, \end{aligned}$$

where we used  $\gamma(2^{[\alpha k]}(t - t_k) - m) = \gamma(2^{[\alpha k]}(t - t_k) - m) \eta_0(2^{[\alpha k]}(t - t_k) - m)$  and (3.6) as before.

It remains to prove (4.6). It suffices to prove

$$\sup_{t_k \in \mathbb{R}} \left\| \mathcal{F}[\eta_0(2^{[\alpha k]}(t - t_k)) \cdot \tilde{u}] \right\|_{X_k} \lesssim \sup_{t_k \in [-T, T]} \left\| \mathcal{F}[\eta_0(2^{[\alpha k]}(t - t_k)) \cdot \tilde{u}] \right\|_{X_k}.$$

For  $t_k > T$ , since  $\tilde{u}$  is supported in  $[-T - \frac{8}{5} \cdot 2^{-[\alpha k] - 5}, T + \frac{8}{5} \cdot 2^{-[\alpha k] - 5}]$ , it is easy to see that

$$\eta_0(2^{[\alpha k]}(t - t_k)) \cdot \tilde{u}(t) = \eta_0(2^{[\alpha k]}(t - T)) \eta_0(2^{[\alpha k]}(t - t_k)) \cdot \tilde{u}(t).$$

Therefore, from (3.6), we obtain

$$\sup_{t_k > T} \left\| \mathcal{F}[\eta_0(2^{[\alpha k]}(t - t_k)) \cdot \tilde{u}] \right\|_{X_k} \lesssim \left\| \mathcal{F}[\eta_0(2^{[\alpha k]}(t - T)) \cdot \tilde{u}] \right\|_{X_k}.$$

A similar argument holds for  $t_k < -T$ , and hence we obtain (4.6) as desired.  $\square$

## 5. STRICHARTZ AND RELATED MULTILINEAR ESTIMATES

Recall the following periodic  $L^4$ - and  $L^6$ -Strichartz estimates due to Bourgain [3]:

$$\|u\|_{L^4_{x,t}(\mathbb{T} \times \mathbb{R})} \lesssim \|u\|_{X^{0, \frac{3}{8}}} \quad \text{and} \quad \|S(t)\phi\|_{L^6_{x,t}(\mathbb{T} \times \mathbb{R})} \leq C_\varepsilon |I|^\varepsilon \|\phi\|_{L^2} \quad (5.1)$$

for any  $\varepsilon > 0$ , where  $\phi$  is a function on  $\mathbb{T}$  such that  $\text{supp } \hat{\phi}$  is contained in an interval  $I$  of length  $|I|$ .

**Lemma 5.1.** *Let  $k, j \in \mathbb{Z}_+$ . Suppose that  $u \in L^2(\mathbb{T} \times \mathbb{R})$  with  $\text{supp } \hat{u} \subset D_{\leq j}$ . Then*

$$\|u\|_{L^4_{x,t}} \lesssim 2^{\frac{3}{8}j} \|u\|_{L^2_{x,t}}. \quad (5.2)$$

Moreover, if, in addition, we assume that  $\text{supp } \hat{u} \subset I \times \mathbb{R}$  for some interval  $I$ , then we have

$$\|u\|_{L^6_{x,t}} \leq C_\varepsilon |I|^\varepsilon 2^{\frac{j}{2}} \|u\|_{L^2_{x,t}} \quad (5.3)$$

for any  $\varepsilon > 0$ .

*Proof.* The  $L^4$ -estimate (5.2) is a direct consequence of (5.1). Writing

$$u(x, t) = c \sum_n \int \hat{u}(n, \tau) e^{inx} e^{it\tau} d\tau = c \sum_n \int \hat{u}(n, \tau + n^2) e^{i(nx+n^2t)} e^{it\tau} d\tau,$$

it follows from (5.1) that

$$\begin{aligned} \|u\|_{L^6_{x,t}} &\lesssim \int \eta_{\leq j}(\tau) \left\| \sum_n \hat{u}(n, \tau + n^2) e^{i(nx+n^2t)} \right\|_{L^6_{x,t}} d\tau \\ &\lesssim |I|^\varepsilon \int \eta_{\leq j}(\tau) \left( \sum_n |\hat{u}(n, \tau + n^2)|^2 \right)^{\frac{1}{2}} d\tau \lesssim |I|^\varepsilon 2^{\frac{j}{2}} \|u\|_{L^2_{x,t}}. \end{aligned}$$

This completes the proof of the  $L^6$ -estimate (5.3).  $\square$

As an immediate consequence of Lemma 5.1, we have the following multilinear estimates.

**Lemma 5.2.** *Let  $u_{k_i, j_i}$  be a function on  $\mathbb{T} \times \mathbb{R}$  such that  $\text{supp } \hat{u}_{k_i, j_i} \subset D_{k_i, \leq j_i}$ ,  $i = 1, \dots, 4$ . Then, we have*

$$\left| \int_{\mathbb{T} \times \mathbb{R}} u_{k_1, j_1} \bar{u}_{k_2, j_2} u_{k_3, j_3} \bar{u}_{k_4, j_4} dx dt \right| \lesssim \prod_{i=1}^4 2^{\frac{3j_i}{8}} \|\mathcal{F}(u_{k_i, j_i})\|_{\ell_n^2 L_\tau^2}, \quad (5.4)$$

$$\left| \int_{\mathbb{T} \times \mathbb{R}} u_{k_1, j_1} \bar{u}_{k_2, j_2} u_{k_3, j_3} \bar{u}_{k_4, j_4} dx dt \right| \lesssim 2^{-\frac{j_1^*}{2}} 2^{\varepsilon k_3^*} \prod_{i=1}^4 2^{\frac{j_i}{2}} \|\mathcal{F}(u_{k_i, j_i})\|_{\ell_n^2 L_\tau^2}, \quad (5.5)$$

for any  $\varepsilon > 0$ . Here,  $j_i^*$  and  $k_i^*$  denote the decreasing rearrangements of  $j_i$  and  $k_i$ .

*Proof.* The first estimate (5.4) follows from  $L^4_{x,t}, L^4_{x,t}, L^4_{x,t}, L^4_{x,t}$ -Hölder inequality and (5.2).

Next, we prove the second estimate (5.5). Without loss of generality, assume  $k_1 \geq k_2 \geq k_3 \geq k_4$ . By writing  $I_{k_1} = \bigcup_{\ell_1} J_{1\ell_1}$  and  $I_{k_2} = \bigcup_{\ell_2} J_{2\ell_2}$ , where  $J_{1\ell_1}$  and  $J_{2\ell_2}$  are intervals of length  $\sim |I_{k_3}|$ , we can decompose  $\hat{u}_{k_i, j_i}$ ,  $i = 1, 2$ , as

$$\hat{u}_{k_i, j_i} = \sum_{\ell_i} \hat{u}_{k_i, j_i, \ell_i},$$

where  $\hat{u}_{k_i, j_i, \ell_i}(n_i, \tau_i) = \mathbf{1}_{J_{i\ell_i}}(n_i) \hat{u}_{k_i, j_i}(n_i, \tau_i)$ . Given  $n_1 \in J_{1\ell_1}$  for some  $\ell_1$ , there exist  $O(1)$  many possible values for  $\ell_2 = \ell_2(\ell_1)$  such that  $n_2 \in J_{2\ell_2}$  under  $n_1 - n_2 + n_3 - n_4 = 0$ . Note that the number of possible values of  $\ell'$  is independent of  $\ell$ .

By  $L^2_{x,t}, L^6_{x,t}, L^6_{x,t}, L^6_{x,t}$ -Hölder inequality (placing the term with the highest modulation in  $L^2_{x,t}$ ) and (5.3), we obtain

$$\begin{aligned} \text{LHS of (5.5)} &= \sum_{\ell_1} \sum_{\ell_2} \sum_{n_1-n_2+n_3-n_4=0} \int_{\tau_1-\tau_2+\tau_3-\tau_4=0} \widehat{u}_{k_1, j_1, \ell_1}(n_1, \tau_1) \\ &\quad \times \widehat{\bar{u}}_{k_2, j_2, \ell_2}(n_2, \tau_2) \widehat{u}_{k_3, j_3}(n_3, \tau_3) \widehat{\bar{u}}_{k_4, j_4}(n_4, \tau_4) d\tau_1 d\tau_2 d\tau_3 \\ &\lesssim \sum_{\ell_1} \sum_{\ell_2} 2^{-\frac{j_1^*}{2}} 2^{\varepsilon k_3^*} \left( \prod_{i=1}^4 2^{\frac{j_i}{2}} \right) \|\widehat{u}_{k_1, j_1, \ell_1}\|_{\ell_n^2 L_\tau^2} \\ &\quad \times \|\widehat{u}_{k_2, j_2, \ell_2}\|_{\ell_n^2 L_\tau^2} \|\widehat{u}_{k_3, j_3}\|_{\ell_n^2 L_\tau^2} \|\widehat{u}_{k_4, j_4}\|_{\ell_n^2 L_\tau^2}. \end{aligned}$$

Then, (5.5) follows from Cauchy-Schwarz inequality in  $\ell_1$ .  $\square$

The proof of (5.5) was based on the  $L^6$ -Strichartz estimate (5.3) after applying Hölder inequality. By refining the analysis, we can obtain the following lemmata, which can be viewed as an refinement of the  $L^6$ -Strichartz estimate (in the multilinear setting) in certain cases.

**Lemma 5.3.** *Let  $\alpha \in [0, 1]$ . Let  $u_i$  be a function on  $\mathbb{T} \times \mathbb{R}$  such that  $\text{supp } \widehat{u}_i \subset D_{k_i, j_i}$ . Assume that  $k_1, k_2, k_3 \geq k_1^* - 5 \geq k_4 + 5$  and  $j_1, j_2, j_3 \geq [\alpha k_1^*]$ . Then, we have*

$$\left| \int_{\mathbb{T} \times \mathbb{R}} u_1 \bar{u}_2 u_3 \bar{u}_4 dx dt \right| \lesssim M \prod_{i=1}^4 2^{\frac{j_i}{2}} \|\mathcal{F}(u_i)\|_{\ell_n^2 L_\tau^2}, \quad (5.6)$$

where  $M$  is given by

$$M = \begin{cases} 2^{-\frac{j_1^*}{2} - \frac{1}{2}\alpha k_1^* + \frac{1}{2}k_4}, & \text{if } j_1^* \neq j_2, \\ 2^{-\frac{j_1^*}{2} - \frac{1}{4}\alpha k_1^* + \frac{1}{2}k_4}, & \text{if } j_1^* = j_2. \end{cases}$$

*Proof.* Letting  $f_i = \widehat{u}_i$  for  $i = 1, 3$  and  $f_i = \widehat{\bar{u}}_i$  for  $i = 2, 4$ , we have

$$\text{LHS of (5.6)} = c \int_{\tau_1-\tau_2+\tau_3-\tau_4=0} \sum_{n_1-n_2+n_3-n_4=0} \prod_{i=1}^4 f_i(n_i, \tau_i) d\tau_1 d\tau_2 d\tau_3. \quad (5.7)$$

• **Case (a):**  $j_4 = j_1^*$ .

By symmetry, assume  $k_1 \geq k_3$ . Then, we can assume  $k_1 \geq k_2^* \geq k_1^* - 3$  under  $n_1 - n_2 + n_3 - n_4 = 0$ . With  $g_i(n, \tau) = f_i(n, \tau + n^2)$ , we have

$$\begin{aligned} (5.7) &\lesssim \int \sum_{n_4} |g_4(n_4, \tau_4)| \sum_{n_1, n_2} |g_1(n_1, \tau_1)| |g_2(n_2, \tau_2)| \\ &\quad \times |g_3(-n_1 + n_2 + n_4, h_3(n_1, n_2, n_4, \tau_1, \tau_2, \tau_4))| d\tau_1 d\tau_2 d\tau_4, \end{aligned}$$

where  $h_3(n_1, n_2, n_4, \tau_1, \tau_2, \tau_4)$  is defined by

$$h_3(n_1, n_2, n_4, \tau_1, \tau_2, \tau_4) = -\tau_1 + \tau_2 + \tau_4 - n_1^2 + n_2^2 + n_4^2 - (-n_1 + n_2 + n_4)^2.$$

For fixed  $n_1, n_4, \tau_1, \tau_2$ , and  $\tau_4$ , define the set  $E_{32} = E_{32}(n_1, n_4, \tau_1, \tau_2, \tau_4)$  by

$$E_{32} = \{n_2 \in \mathbb{Z} : h_3(n_1, n_2, n_4, \tau_1, \tau_2, \tau_4) = O(2^{j_3})\}.$$

Noting that  $|n_1 - n_4| \sim 2^{k_1^*}$  and that

$$-n_1^2 + n_2^2 + n_4^2 - (-n_1 + n_2 + n_4)^2 = 2n_2(n_1 - n_4) - 2n_1^2 + 2n_1n_4,$$

we conclude that

$$|E_{32}| \lesssim 1 + 2^{j_3 - k_1^*}. \quad (5.8)$$

Then, by Cauchy-Schwarz inequality in  $n_2, n_1, n_4$ , we obtain

$$\begin{aligned} (5.7) &\lesssim (1 + 2^{j_3 - k_1^*})^{\frac{1}{2}} \int \sum_{n_4} |g_4(n_4, \tau_4)| \sum_{n_1} |g_1(n_1, \tau_1)| \left( \sum_{n_2} |g_2(n_2, \tau_2)|^2 \right. \\ &\quad \times \left. |g_3(-n_1 + n_2 + n_4, h_3(n_1, n_2, n_4, \tau_1, \tau_2, \tau_4))|^2 \right)^{\frac{1}{2}} d\tau_1 d\tau_2 d\tau_4 \\ &\lesssim (1 + 2^{j_3 - k_1^*})^{\frac{1}{2}} 2^{\frac{k_4}{2}} \|g_4(n_4, \tau_4)\|_{\ell_{n_4}^2 L_{\tau_4}^2} \sup_{n_4} \int \|g_1(n_1, \tau_1)\|_{\ell_{n_1}^2} \left( \sum_{n_2} |g_2(n_2, \tau_2)|^2 \right. \\ &\quad \times \left. \sum_{n_1} \|g_3(-n_1 + n_2 + n_4, h_3(n_1, n_2, n_4, \tau_1, \tau_2, \tau_4))\|_{L_{\tau_4}^2}^2 \right)^{\frac{1}{2}} d\tau_1 d\tau_2 \end{aligned}$$

Noting that  $h_3$  is linear in  $\tau_4$  and applying Cauchy-Schwarz inequality in  $\tau_1$  and  $\tau_2$ ,

$$\begin{aligned} &\leq (1 + 2^{j_3 - k_1^*})^{\frac{1}{2}} 2^{\frac{k_4}{2}} \left( \prod_{i_1=1}^2 \|g_{i_1}\|_{L_{\tau_{i_1}}^1 \ell_{n_{i_1}}^2} \right) \left( \prod_{i_2=3}^4 \|g_{i_2}\|_{\ell_{n_{i_2}}^2 L_{\tau_{i_2}}^2} \right) \\ &\lesssim 2^{-\frac{j_1^*}{2}} (1 + 2^{j_3 - k_1^*})^{\frac{1}{2}} 2^{-\frac{j_3}{2}} \cdot 2^{\frac{k_4}{2}} \prod_{i=1}^4 2^{\frac{j_i}{2}} \|\mathcal{F}(u_i)\|_{\ell_n^2 L_\tau^2}, \end{aligned}$$

yielding (5.6).

- **Case (b):**  $j_1 = j_1^*$ . (The case  $j_3 = j_1^*$  follows in the same manner by symmetry.)

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} (5.7) &\lesssim \int \sum_{n_4} |g_4(n_4, \tau_4)| \sum_{n_1, n_3} \int |g_1(n_1, \tau_1)| |g_3(n_3, \tau_3)| \\ &\quad \times |g_2(n_1 + n_3 - n_4, h_2(n_1, n_3, n_4, \tau_1, \tau_3, \tau_4))| d\tau_1 d\tau_3 d\tau_4 \end{aligned}$$

where  $h_2(n_1, n_3, n_4, \tau_1, \tau_3, \tau_4)$  is defined by

$$h_2(n_1, n_3, n_4, \tau_1, \tau_3, \tau_4) = \tau_1 + \tau_3 - \tau_4 + n_1^2 + n_3^2 - n_4^2 - (n_1 + n_3 - n_4)^2.$$

For fixed  $n_1, n_4, \tau_1, \tau_3$ , and  $\tau_4$ , define the set  $E_{23} = E_{23}(n_1, n_4, \tau_1, \tau_3, \tau_4)$  by

$$E_{23} = \{n_3 \in \mathbb{Z} : h_2(n_1, n_3, n_4, \tau_1, \tau_3, \tau_4) = O(2^{j_2})\}.$$

Then, from  $n_1^2 + n_3^2 - n_4^2 - (n_1 + n_3 - n_4)^2 = -2n_3(n_1 - n_4) + 2n_1n_4 - 2n_4^2$  and  $|n_1 - n_4| \sim 2^{k_1^*}$ , we conclude that  $|E_{23}| \lesssim 1 + 2^{j_2 - k_1^*}$ . Then, proceeding as before with Cauchy-Schwarz inequality and noting that  $h_2$  is linear in  $\tau_1$ , we have

$$\begin{aligned} (5.7) &\lesssim (1 + 2^{j_2 - k_1^*})^{\frac{1}{2}} 2^{\frac{k_4}{2}} \int \|g_4(n_4, \tau_4)\|_{\ell_{n_4}^2} \|g_1(n_1, \tau_1)\|_{\ell_{n_1}^2 L_{\tau_1}^2} \sup_{n_4} \left( \sum_{n_3} |g_3(n_3, \tau_3)|^2 \right. \\ &\quad \times \left. \sum_{n_1} \|g_2(n_1 + n_3 - n_4, h_2(n_1, n_3, n_4, \tau_1, \tau_3, \tau_4))\|_{L_{\tau_1}^2}^2 \right)^{\frac{1}{2}} d\tau_3 d\tau_4 \\ &\lesssim 2^{-\frac{j_1^*}{2}} (1 + 2^{j_2 - k_1^*})^{\frac{1}{2}} 2^{-\frac{j_2}{2}} \cdot 2^{\frac{k_4}{2}} \prod_{i=1}^4 2^{\frac{j_i}{2}} \|\mathcal{F}(u_i)\|_{\ell_n^2 L_\tau^2}. \end{aligned}$$

• **Case (c):**  $j_2 = j_1^*$ .

In this case, we do not use the multilinear argument. The desired estimate follows from  $L_{x,t}^4, L_{x,t}^2, L_{x,t}^4, L_{x,t}^\infty$ -Hölder inequality and the  $L^4$ -Strichartz estimate (5.2).  $\square$

**Remark 5.4.** In the periodic case, we have a lower bound  $|E| \geq 1$  unless  $|E| = 0$ . See (5.8). Hence, our estimates are worse than those in the non-periodic setting. Compare this with Propositions 4.1 and 4.2 in [13].

**Lemma 5.5.** *Let  $\alpha \in [0, 1]$ . Let  $u_i$  be a function on  $\mathbb{T} \times \mathbb{R}$  such that  $\text{supp } \widehat{u}_i \subset D_{k_i, j_i}$ . Assume that  $k_3, k_4 \leq k_2^* - 10$ ,  $j_1, j_2, j_3 \geq [\alpha k_1^*]$ , and  $j_1^* \geq |\Phi(\bar{n})|$ , where  $\Phi(\bar{n})$  is defined in (2.5). Then, we have*

$$\left| \int_{\mathbb{T} \times \mathbb{R}} u_1 \bar{u}_2 u_3 \bar{u}_4 dx dt \right| \lesssim M \prod_{i=1}^4 2^{\frac{j_i}{2}} \|\mathcal{F}(u_i)\|_{\ell_n^2 L_t^2}, \quad (5.9)$$

where  $M$  is given by

$$M = \begin{cases} 2^{-\frac{1}{2}(1+\alpha)k_1^*}, & \text{if } |k_3 - k_4| \geq 2, \\ 2^{-\frac{1}{2}(1+\alpha)k_1^* + \frac{k_3^*}{4}}, & \text{otherwise.} \end{cases}$$

Moreover, when  $j_3 = j_1^*$  and  $k_3 \geq k_4 + 2$ , (5.9) holds with

$$M = 2^{-\frac{k_1^*}{2} - \frac{k_3}{2} + \frac{k_4}{2} - \frac{\beta}{2}}, \quad \text{where } \beta = \min(\alpha k_1^*, k_3). \quad (5.10)$$

*Proof.* First, we consider the case  $|k_3 - k_4| \geq 2$ . Then, we have  $j_1^* \geq k_1^* + k_3^* - 5$  since  $|\Phi(\bar{n})| \sim |(n_2 - n_3)(n_3 - n_4)| \sim n_1^* |n_3 - n_4| \sim n_1^* n_3^*$ . When (a):  $j_4 = j_1^*$  or (b):  $j_1 = j_1^*$ , the desired estimate follows from the corresponding cases in the proof of Lemma 5.3.

Next, consider the case (c)  $j_2 = j_1^*$ . With the notations from the proof of Lemma 5.3, we have

$$\begin{aligned} \text{LHS of (5.9)} &\lesssim \int \sum_{n_4} |g_4(n_4, \tau_4)| \sum_{n_2, n_3} |g_2(n_2, \tau_2)| |g_3(n_3, \tau_3)| \\ &\quad \times |g_1(n_2 - n_3 + n_4, h_1(n_2, n_3, n_4, \tau_2, \tau_3, \tau_4))| d\tau_2 d\tau_3 d\tau_4, \end{aligned} \quad (5.11)$$

where  $h_1(n_2, n_3, n_4, \tau_2, \tau_3, \tau_4)$  is defined by

$$h_1(n_2, n_3, n_4, \tau_2, \tau_3, \tau_4) = \tau_2 - \tau_3 + \tau_4 + n_2^2 - n_3^2 + n_4^2 - (n_2 - n_3 + n_4)^2. \quad (5.12)$$

For fixed  $n_2, n_4, \tau_2, \tau_3$ , and  $\tau_4$ , define the set  $E_{13} = E_{13}(n_2, n_3, \tau_2, \tau_3, \tau_4)$  by

$$E_{13} = \{n_3 \in \mathbb{Z} : h_1(n_2, n_3, n_4, \tau_2, \tau_3, \tau_4) = O(2^{j_1})\}.$$

Then, by writing  $n_2^2 - n_3^2 + n_4^2 - (n_2 - n_3 + n_4)^2 = -2n_3^2 + 2(n_2 + n_4)n_3 - 2n_2n_4$ , we have  $|\partial_{n_3} h_1(n_3)| = |2(n_2 - 2n_3 + n_4)| \sim 2^{k_1^*}$  since  $|n_2| \gg |n_3|, |n_4|$ . Hence, we conclude that  $|E_{13}| \lesssim 1 + 2^{j_1 - k_1^*}$ . Proceeding as in the proof of Lemma 5.3, we obtain

$$(5.11) \lesssim 2^{-\frac{j_1^*}{2}} (1 + 2^{j_1 - k_1^*})^{\frac{1}{2}} 2^{-\frac{j_1}{2}} \cdot 2^{\frac{k_4}{2}} \prod_{i=1}^4 2^{\frac{j_i}{2}} \|\mathcal{F}(u_i)\|_{\ell_n^2 L_t^2}.$$

Hence, it remains to consider the case  $j_3 = j_1^*$ . With  $g_i(n, \tau) = f_i(n, \tau + n^2)$ , we have

$$\begin{aligned} \text{LHS of (5.9)} &\lesssim \int \sum_{n_3} |g_3(n_3, \tau_3)| \sum_{n_2, n_4} |g_2(n_2, \tau_2)| |g_4(n_4, \tau_4)| \\ &\quad \times |g_1(n_2 - n_3 + n_4, h_1(n_2, n_3, n_4, \tau_2, \tau_3, \tau_4))| d\tau_2 d\tau_3 d\tau_4, \end{aligned} \quad (5.13)$$



where  $h_1(n_2, n_3, n_4, \tau_2, \tau_3, \tau_4)$  is as in (5.12). For fixed  $n_2, n_3, \tau_2, \tau_3$ , and  $\tau_4$ , define the set  $E_{14} = E_{14}(n_2, n_3, \tau_2, \tau_3, \tau_4)$  by

$$E_{14} = \{n_4 \in \mathbb{Z} : h_1(n_2, n_3, n_4, \tau_2, \tau_3, \tau_4) = O(2^{j_1})\}.$$

Then, from  $n_2^2 - n_3^2 + n_4^2 - (n_2 - n_3 + n_4)^2 = -2n_4(n_2 - n_3) - 2n_3^2 + 2n_2n_3$  and  $|n_2 - n_3| \sim 2^{k_1^*}$ , we conclude that  $|E_{14}| \lesssim 1 + 2^{j_1 - k_1^*}$ . Proceeding as in the proof of Lemma 5.3, we have

$$\begin{aligned} (5.13) &\lesssim 2^{-\frac{j_1}{2}} (1 + 2^{j_1 - k_1^*})^{\frac{1}{2}} 2^{-\frac{j_1}{2}} \cdot 2^{\frac{k_3}{2}} \prod_{i=1}^4 2^{\frac{j_i}{2}} \|\mathcal{F}(u_i)\|_{\ell_n^2 L_\tau^2} \\ &\lesssim 2^{-\frac{1}{2}(1+\alpha)k_1^*} \prod_{i=1}^4 2^{\frac{j_i}{2}} \|\mathcal{F}(u_i)\|_{\ell_n^2 L_\tau^2}. \end{aligned}$$

Next, we consider the case  $|k_3 - k_4| \leq 1$ . We separate the argument into two cases: (i)  $|n_3 - n_4| \gtrsim 2^{\frac{k_3^*}{2}}$  and (ii)  $|n_3 - n_4| \ll 2^{\frac{k_3^*}{2}}$ . In Case (i), we have  $j_1^* \geq k_1^* + \frac{k_3^*}{2} - 5$ . Hence, by repeating the previous argument, we obtain

$$\text{LHS of (5.9)} \lesssim 2^{-\frac{1}{2}(1+\alpha)k_1^*} 2^{\frac{k_3^*}{4}} \prod_{i=1}^4 2^{\frac{j_i}{2}} \|\mathcal{F}(u_i)\|_{\ell_n^2 L_\tau^2}. \quad (5.14)$$

In Case (ii), we write  $I_{k_i} = \bigcup_{\ell_i} J_{i\ell_i}$ ,  $i = 3, 4$ , where  $|J_{i\ell_i}| = 2^{\frac{k_i^*}{2}}$ . As in the proof of Lemma 5.2, it follows that, if  $n_3 \in J_{3\ell_3}$  for some  $\ell_3$ , there are  $O(1)$  many possible values for  $\ell_4 = \ell_4(\ell_3)$  such that  $n_4 \in J_{4\ell_4}$ . Then, by writing

$$\sum_{n_3} \sum_{n_4} = \sum_{\ell_3} \sum_{\ell_4 = \ell_4(\ell_3)} \sum_{n_3 \in J_{3\ell_3}} \sum_{n_4 \in J_{4\ell_4}}$$

and repeating the previous argument for each  $\ell_3$ , we only lose  $|J_{i\ell_i}| = 2^{\frac{k_3^*}{4}}$  by applying Cauchy-Schwarz inequality in  $n_3$  or  $n_4$  at the end. Finally, applying Cauchy-Schwarz inequality in  $\ell_3$ , we obtain (5.14).

The second claim with (5.10) follows from Case (a) in the proof of Lemma 5.3 by switching the indices  $1 \leftrightarrow 3$ . In this case, we have  $|E_{12}| \lesssim 1 + 2^{j_1 - k_3}$  and it suffices to note that  $j_3 \geq k_1^* + k_3 - 5$  and  $(1 + 2^{j_1 - k_3})^{\frac{1}{2}} 2^{-\frac{j_1}{2}} \lesssim 2^{-\frac{1}{2} \min(j_1, k_3)} \lesssim 2^{-\frac{1}{2} \min(\alpha k_1^*, k_3)}$ .  $\square$

**Remark 5.6.** It follows from the proof of Lemma 5.5 that we lose  $2^{\frac{k_3}{2}}$  in the case  $j_3 = j_1^*$ , while we lose  $2^{\frac{k_4}{2}}$  in other cases.

## 6. TRILINEAR ESTIMATES

In this section, we prove the main trilinear estimates on the cubic nonlinear terms  $\mathcal{N}$  and  $\mathcal{R}$  in (2.1) and (2.2) of the Wick ordered cubic NLS (1.3).

**Proposition 6.1.** *Let  $s \in (-\frac{1}{8}, 0)$  and  $T \in (0, 1]$ . Then, with  $\alpha = -4s+$ , there exists  $\theta > 0$  such that*

$$\|\mathcal{N}(u_1, u_2, u_3)\|_{N^{s, \alpha}(T)} \lesssim T^\theta \prod_{i=1}^3 \|u_i\|_{F^{s, \alpha}(T)}, \quad (6.1)$$

$$\|\mathcal{R}(u_1, u_2, u_3)\|_{N^{s, \alpha}(T)} \lesssim T^\theta \prod_{i=1}^3 \|u_i\|_{F^{s, \alpha}(T)}, \quad (6.2)$$

where  $\mathcal{N}(u_1, u_2, u_3)$  and  $\mathcal{R}(u_1, u_2, u_3)$  are defined in (2.1) and (2.2).

In the following, we first prove the preliminary lemmata and then present the proof of Proposition 6.1 at the end of this section. Let  $\Phi(\bar{n}) = 2(n - n_1)(n - n_3)$  be the phase function as in (2.5). Then, under  $\{n_1, n_3\} \neq \{n, n_2\}$ , we have the following.

**Support conditions:**

(i) If  $|n| \sim |n_3| \gg |n_1|, |n_2|$ , then

$$|\Phi(\bar{n})| \sim n_1^* |n_2 - n_1|. \quad (6.3)$$

(ii) If  $|n| \sim |n_2| \gg |n_1|, |n_3|$ , then

$$|\Phi(\bar{n})| \sim (n_1^*)^2. \quad (6.4)$$

(iii) If  $|n| \sim |n_2| \sim |n_3| \gg |n_1|$ , then

$$|\Phi(\bar{n})| \sim (n_1^*)^2. \quad (6.5)$$

(iv) If  $|\Phi(\bar{n})| \ll n_1^*$ , then

$$|n| \sim |n_1| \sim |n_2| \sim |n_3|.$$

The conditions (i)–(iii) hold under the symmetries  $n_1 \leftrightarrow n_3$  and  $n \leftrightarrow n_2$ , and  $\{n_1, n_3\} \leftrightarrow \{n, n_2\}$ , respectively. In the following, we assume that  $u_i$  is a function on  $\mathbb{T} \times \mathbb{R}$  such that  $\text{supp } \widehat{u}_i \subset I_{k_i} \times \mathbb{R}$ ,  $i = 1, 2, 3$ .

**Lemma 6.2** (high  $\times$  low  $\times$  low  $\rightarrow$  high). *Let  $\alpha \geq 0$ . If  $k_4 \geq 20$ ,  $|k_1 - k_4| \leq 5$ , and  $k_2, k_3 \leq k_1 - 10$ , then we have*

$$\|\mathbf{P}_{k_4} \mathcal{N}(u_1, u_2, u_3)\|_{N_{k_4}^\alpha} \lesssim \min(2^{-\frac{1}{2}(\alpha-\varepsilon)k_1^*}, 2^{-\frac{1}{2}(1-\varepsilon)k_1^*}) \prod_{i=1}^3 \|u_i\|_{F_{k_i}^\alpha} \quad (6.6)$$

for any  $\varepsilon > 0$ . The estimate (6.6) holds under permutation of indices  $k_1, k_2$ , and  $k_3$ .

*Proof.* Let  $\gamma : \mathbb{R} \rightarrow [0, 1]$  be a smooth cutoff function supported on  $[-1, 1]$  with  $\gamma \equiv 1$  on  $[-\frac{1}{4}, \frac{1}{4}]$  such that

$$\sum_m \gamma^3(t - m) \equiv 1, \quad t \in \mathbb{R}.$$

Then, there exist  $c, C > 0$  such that

$$\eta_0(2^{[\alpha k_4]}(t - t_{k_4})) = \eta_0(2^{[\alpha k_4]}(t - t_{k_4})) \sum_{|m| \leq C} \gamma^3(2^{[\alpha k_4]+c}(t - t_{k_4}) - m)$$

and

$$\eta_0(2^{[\alpha k_i]}t) \cdot \gamma(2^{[\alpha k_4]+c}t) = \gamma(2^{[\alpha k_4]+c}t) \quad (6.7)$$

for  $i = 1, 2, 3$ . Then, from the definition and Lemma 3.1, the left-hand side of (6.6) is estimated by

$$\begin{aligned} & C \sup_{t_{k_4} \in \mathbb{R}} \left\| (\tau - n^2 + i2^{[\alpha k_4]})^{-1} \mathbf{1}_{I_{k_4}}(n) \mathcal{F}[\gamma(2^{[\alpha k_4]+c}(t - t_{k_4}) - m) \cdot u_1] \right. \\ & \quad \left. * \mathcal{F}[\gamma(2^{[\alpha k_4]+c}(t - t_{k_4}) - m) \cdot \bar{u}_2] * \mathcal{F}[\gamma(2^{[\alpha k_4]+c}(t - t_{k_4}) - m) \cdot u_3] \right\|_{X_{k_4}}. \end{aligned} \quad (6.8)$$

Let  $f_{k_i} = \mathcal{F}[\gamma(2^{[\alpha k_4]+c}(t - t_{k_4})) \cdot u_i]$ ,  $i = 1, 3$ , and  $\overline{f_{k_2}} = \mathcal{F}[\gamma(2^{[\alpha k_4]+c}(t - t_{k_4})) \cdot u_2]$ . Then, from the definition (3.2) of  $X_k$ , we have

$$(6.8) \lesssim \sup_{t_{k_4} \in \mathbb{R}} \sum_{j_4=0}^{\infty} 2^{\frac{j_4}{2}} \sum_{j_1, j_2, j_3 \geq [\alpha k_4]} \|(2^{j_4} + 2^{[\alpha k_4]})^{-1} \times \mathbf{1}_{D_{k_4, j_4}} \cdot (f_{k_1, j_1} * \widetilde{f_{k_2, j_2}} * f_{k_3, j_3})\|_{\ell_n^2 L_\tau^2}, \quad (6.9)$$

where  $\widetilde{f}(n, \tau) = f(-n, -\tau)$  and  $f_{k_i, j_i}$ ,  $i = 1, 2, 3$ , is defined by

$$f_{k_i, j_i}(n, \tau) = \begin{cases} f_{k_i}(n, \tau) \eta_{j_i}(\tau - n^2), & \text{for } j_i > [\alpha k_4], \\ f_{k_i}(n, \tau) \eta_{\leq [\alpha k_4]}(\tau - n^2), & \text{for } j_i = [\alpha k_4]. \end{cases}$$

Using the fact  $\mathbf{1}_{D_{k_4, j_4}} \leq \mathbf{1}_{D_{k_4, \leq j_4}}$ , we have

$$\begin{aligned} (6.9) &\lesssim \sup_{t_{k_4} \in \mathbb{R}} \left( \sum_{j_4 < [\alpha k_4]} + \sum_{j_4 \geq [\alpha k_4]} \right) 2^{\frac{j_4}{2}} \sum_{j_1, j_2, j_3 \geq [\alpha k_4]} \|(2^{j_4} + 2^{[\alpha k_4]})^{-1} \mathbf{1}_{D_{k_4, j_4}} \\ &\quad \times f_{k_1, j_1} * \widetilde{f_{k_2, j_2}} * f_{k_3, j_3}\|_{\ell_n^2 L_\tau^2} \\ &\lesssim \sup_{t_{k_4} \in \mathbb{R}} \sum_{j_1, j_2, j_3, j_4 \geq [\alpha k_4]} 2^{-\frac{j_4}{2}} \|\mathbf{1}_{D_{k_4, \leq j_4}} \cdot (f_{k_1, j_1} * \widetilde{f_{k_2, j_2}} * f_{k_3, j_3})\|_{\ell_n^2 L_\tau^2} \\ &\lesssim \sup_{t_{k_4} \in \mathbb{R}} \sup_{j_4 \geq [\alpha k_4]} \sum_{j_1, j_2, j_3 \geq [\alpha k_4]} 2^{-(\frac{1}{2}-)j_4} \|\mathbf{1}_{D_{k_4, \leq j_4}} \cdot (f_{k_1, j_1} * \widetilde{f_{k_2, j_2}} * f_{k_3, j_3})\|_{\ell_n^2 L_\tau^2}. \quad (6.10) \end{aligned}$$

From duality and Lemma 5.2 along with the support conditions (6.3) and (6.4), we have

$$(6.10) \lesssim \sup_{t_{k_4} \in \mathbb{R}} \min(2^{-\frac{1}{2}(\alpha-\varepsilon)k_1^*}, 2^{-\frac{1}{2}(1-\varepsilon)k_1^*}) \sum_{j_1, j_2, j_3 \geq [\alpha k_4]} \prod_{i=1}^3 2^{\frac{j_i}{2}} \|f_{k_i, j_i}\|_{\ell_n^2 L_\tau^2}.$$

Finally, by applying (3.5) with (6.7), we obtain (6.6). When  $k_2 \geq k_2^*$ , (6.4) actually provides a better estimate with  $2^{-(1-\varepsilon)k_1^*}$ .  $\square$

In the proof of Lemma 6.2, we carried the supremum over  $t_{k_4}$  along the argument. For simplicity, we may drop the supremum over  $t_{k_4}$  in the following proofs without explicitly stating so.

**Lemma 6.3** (high  $\times$  high  $\times$  low  $\rightarrow$  high). *Let  $\alpha \geq 0$ . If  $k_4 \geq 20$ ,  $|k_i - k_4| \leq 5$ ,  $i = 1, 2$ , and  $k_3 \leq k_1 - 10$ , then we have*

$$\|\mathbf{P}_{k_4} \mathcal{N}(u_1, u_2, u_3)\|_{N_{k_4}^\alpha} \lesssim 2^{-(1-\varepsilon)k_1^*} \|u_1\|_{F_{k_1}^\alpha} \|u_2\|_{F_{k_2}^\alpha} \|u_3\|_{F_{k_3}^\alpha} \quad (6.11)$$

for any  $\varepsilon > 0$ . The estimate (6.11) holds under permutation of indices  $k_1, k_2$ , and  $k_3$ .

*Proof.* We proceed as in Lemma 6.2. In this case, we have  $j_1^* \geq 2k_1^* - 5$  from the support condition (6.5). Then, (6.11) follows from (5.5) in Lemma 5.2.  $\square$

**Lemma 6.4** (high  $\times$  high  $\times$  high  $\rightarrow$  high). *Let  $\alpha \geq 0$ . If  $k_4 \geq 20$ ,  $|k_i - k_4| \leq 5$ ,  $i = 1, 2, 3$ , then we have*

$$\begin{aligned} \|\mathbf{P}_{k_4} \mathcal{N}(u_1, u_2, u_3)\|_{N_{k_4}^\alpha} + \|\mathbf{P}_{k_4} \mathcal{R}(u_1, u_2, u_3)\|_{N_{k_4}^\alpha} \\ \lesssim 2^{-\frac{1}{2}(\alpha-\varepsilon)k_1^*} \|u_1\|_{F_{k_1}^\alpha} \|u_2\|_{F_{k_2}^\alpha} \|u_3\|_{F_{k_3}^\alpha} \end{aligned} \quad (6.12)$$

for any  $\varepsilon > 0$ .

*Proof.* In this case, we do not have any lower bound on the size of the phase function  $\Phi(\bar{n})$ . Proceeding as in Lemma 6.2 with  $j_i \geq [\alpha k_4]$ ,  $i = 1, \dots, 4$ , (6.12) follows from (5.4) in Lemma 5.2.  $\square$

**Lemma 6.5** (high  $\times$  high  $\times$  high  $\rightarrow$  low). *Let  $\alpha \geq 0$ . If  $k_3 \geq 20$ ,  $|k_3 - k_i| \leq 5$ ,  $i = 1, 2$ , and  $k_4 \leq k_1 - 30$ , then we have*

$$\|\mathbf{P}_{k_4} \mathcal{N}(u_1, u_2, u_3)\|_{N_{k_4}^\alpha} \lesssim \min(M_1, M_2) \|u_1\|_{F_{k_1}^\alpha} \|u_2\|_{F_{k_2}^\alpha} \|u_3\|_{F_{k_3}^\alpha} \quad (6.13)$$

for any  $\varepsilon > 0$ , where  $M_1$  and  $M_2$  are given by

$$M_1 = 2^{-(1-\alpha-\varepsilon)k_1^* - \alpha k_4} \quad \text{and} \quad M_2 = 2^{-(1-\frac{3}{4}\alpha-\varepsilon)k_1^* + (\frac{1}{2}-\alpha)k_4}.$$

*Proof.* Let  $\gamma : \mathbb{R} \rightarrow [0, 1]$  be as in the proof of Lemma 6.2. Then, there exist  $c, C > 0$  such that

$$\eta_0(2^{[\alpha k_4]}(t - t_{k_4})) = \eta_0(2^{[\alpha k_4]}(t - t_{k_4})) \sum_{|m| \leq C2^{[\alpha k_1] - [\alpha k_4]}} \gamma^3(2^{[\alpha k_1] + c}(t - t_{k_4}) - m)$$

and  $\eta_0(2^{[\alpha k_i]}t) \cdot \gamma(2^{[\alpha k_1] + c}t) = \gamma(2^{[\alpha k_1] + c}t)$  for  $i = 1, 2, 3$ . By Lemma 3.1, the left-hand side of (6.13) is estimated by

$$C \sup_{t_{k_4} \in \mathbb{R}} \left\| (\tau - n^2 + i2^{[\alpha k_4]})^{-1} \mathbf{1}_{I_{k_4}}(n) \sum_{|m| \leq C2^{[\alpha k_1] - [\alpha k_4]}} F_{m, t_{k_4}}(n, \tau) \right\|_{X_{k_4}},$$

where  $F_{m, t_{k_4}}(n, \tau) = \widehat{v}_{1, m} * \widehat{v}_{2, m} * \widehat{v}_{3, m}(n, \tau)$  and  $v_{i, m} = \gamma(2^{[\alpha k_1] + c}(t - t_{k_4}) - m) \cdot u_i$ ,  $i = 1, 2, 3$ . Proceeding as in the proof of Lemma 6.2 with (3.5), it suffices to prove that

$$2^{\alpha(k_1 - k_4)} \sum_{j_4 \geq [\alpha k_4]} 2^{-\frac{j_4}{2}} \| \mathbf{1}_{\widetilde{D}_{k_4, j_4}} \cdot (f_{k_1, j_1} * \widetilde{f}_{k_2, j_2} * f_{k_3, j_3}) \|_{\ell_n^2 L_\tau^2} \lesssim M \prod_{i=1}^3 2^{\frac{j_i}{2}} \|f_{k_i, j_i}\|_{\ell_n^2 L_\tau^2}$$

for any  $f_{k_i, j_i} : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}_+$  supported on  $\widetilde{D}_{k_i, j_i}$  with  $j_i \geq [\alpha k_1]$ ,  $i = 1, 2, 3$ , where

$$\widetilde{D}_{k_i, j_i} = \begin{cases} D_{k_i, \leq j_i}, & \text{when } j_i = [\alpha k_1], \\ D_{k_i, j_i}, & \text{when } j_i > [\alpha k_1]. \end{cases}$$

Here, we can assume that  $j_i \geq [\alpha k_1]$ ,  $i = 1, 2, 3$ , thanks to the time localization over an interval of size  $\sim 2^{-[\alpha k_1] - c}$  and (3.5). Hence, (6.13) with  $M_1$  follows from (5.5) in Lemma 5.2 with (6.5):  $j_1^* \geq 2k_1^* - 5$ , while (6.13) with  $M_2$  follows from Lemma 5.3.  $\square$

**Lemma 6.6** (high  $\times$  high  $\times$  low  $\rightarrow$  low). *Let  $\alpha \in [0, 1]$ . If  $k_1 \geq 20$ ,  $|k_1 - k_2| \leq 5$ , and  $k_3, k_4 \leq k_1 - 10$ , then we have*

$$\|\mathbf{P}_{k_4} \mathcal{N}(u_1, u_2, u_3)\|_{N_{k_4}^\alpha} \lesssim \min(M_1, M_2) \|u_1\|_{F_{k_1}^\alpha} \|u_2\|_{F_{k_2}^\alpha} \|u_3\|_{F_{k_3}^\alpha}, \quad (6.14)$$

where  $M_j$ ,  $j = 1, 2$ , is given by

$$M_1 = 2^{(-\frac{1}{2} + \alpha + \varepsilon)k_1^* - \alpha k_4 - \beta_1} \quad \text{with } \beta_1 = \begin{cases} \frac{k_3^*}{2}, & \text{if } |k_3 - k_4| \geq 2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$M_2 = \begin{cases} 2^{(-\frac{1}{2} + \frac{1}{2}\alpha + \varepsilon)k_1^* - \alpha k_4}, & \text{if } |k_3 - k_4| \geq 2, \\ 2^{(-\frac{1}{2} + \frac{1}{2}\alpha + \varepsilon)k_1^* + \frac{k_3^*}{4} - \alpha k_4}, & \text{otherwise,} \end{cases}$$

for any  $\varepsilon > 0$ .

Moreover, when  $j_3 = j_1^*$  and  $k_3 \geq k_4 + 2$  (6.14) holds with the constant given by

$$M_3 = \min \left( 2^{-\frac{k_1^*}{2} + \frac{\alpha}{2}k_3 + (\frac{1}{2} - \alpha + \varepsilon)k_4}, 2^{(-\frac{1}{2} + \alpha)k_1^* - \frac{k_3}{2} + (\frac{1}{2} - \alpha)k_4 - \frac{\min(\alpha k_1^*, k_3)}{2}} \right). \quad (6.15)$$

While the estimate (6.14) holds under permutation of indices  $k_1$ ,  $k_2$ , and  $k_3$ , we have a better estimate when  $k_1, k_3 \geq k_2 + 10$  in view of (6.4).

*Proof.* Proceeding as in the proof of Lemma 6.5, (6.14) with  $M_1$  follows from (5.5) in Lemma 5.2 with the support condition (6.3) and (6.4). Note that we have  $j_1^* \geq k_1^* + k_3^* - 5$  when  $|k_3 - k_4| \geq 2$ . Similarly, (6.14) with  $M_2$  follows from Lemma 5.5.

The argument above suffices to prove the main trilinear estimate (6.1) for  $s > -\frac{1}{8}$  (see Summary (v) below) except for the case  $j_3 = j_1^*$  and  $k_3 \geq k_4 + 2$ , where the argument above is good only for  $s > -\frac{1}{10}$ . See Remark 5.6 and Summary (v) below. In order to go below  $s = -\frac{1}{10}$ , we need a different argument when  $j_3 = j_1^*$  and  $k_3 \geq k_4 + 2$ .

Note that we have  $j_3 \geq k_1^* + k_3 - 5$  from (6.3). In this case, we do not perform time-localization corresponding to the highest (spatial) frequency. Instead, we need to perform time-localization over time intervals of size  $\sim 2^{-[\alpha k_3]}$ . Write  $v_3 = \eta_0(2^{[\alpha k_4]}(t - t_{k_4})) \cdot u_3$  as a sum of  $O(2^{\alpha(k_3 - k_4)})$  many terms localized over time intervals of size  $\sim 2^{-[\alpha k_3]}$ . Then, by Hölder and Hausdorff-Young's inequalities with Lemmata 3.2 and 3.7 (a), we have

$$\begin{aligned} \text{LHS of (6.14)} &\sim \sum_{j_4 \geq [\alpha k_4]} 2^{-\frac{j_4}{2}} \|\mathbf{1}_{D_{k_4, \leq j_4}} \cdot (\widehat{u}_1 * \mathcal{F}(\overline{u}_2) * \widehat{v}_3)\|_{\ell_n^2 L_\tau^2} \\ &\lesssim 2^{(\frac{1}{2} - \frac{\alpha}{2} + \varepsilon)k_4} \|u_1\|_{L_t^\infty L_x^2} \|u_2\|_{L_t^\infty L_x^2} \|v_3\|_{L_t^2 L_x^\infty} \\ &\lesssim 2^{-\frac{j_3}{2}} 2^{\frac{k_3}{2}} 2^{(\frac{1}{2} - \frac{1}{2}\alpha + \varepsilon)k_4} \|u_1\|_{F_{k_1}^\alpha} \|u_2\|_{F_{k_2}^\alpha} \|v_3\|_{F_{k_3}^\alpha} \\ &\lesssim 2^{-\frac{k_1^*}{2} + \frac{\alpha}{2}k_3 + (\frac{1}{2} - \alpha + \varepsilon)k_4} \|u_1\|_{F_{k_1}^\alpha} \|u_2\|_{F_{k_2}^\alpha} \|u_3\|_{F_{k_3}^\alpha}, \end{aligned} \quad (6.16)$$

yielding the first bound in (6.15).

Note that the time localization step affects the modulation. In particular, that the modulation of  $v_3$  may be much smaller than  $2^{j_1^*}$ . Hence, strictly speaking, we first need to decompose  $v_3$  as  $v_3 = v_{3,+} + v_{3,-}$ , where  $v_{3,+}$  is the high modulation piece defined by  $v_{3,+} = \mathcal{F}^{-1}[\eta_{\geq k_1^* + k_3 - 10}(\tau - n^2)\widehat{v}_3]$  and  $v_{3,-} = v_3 - v_{3,+}$ . Then, apply the argument in (6.16) to  $v_{3,+}$ . As for  $v_{3,-}$ , we apply the time localization  $\gamma(2^{[\alpha k_1] + c}(t - t_{k_4}) - m)$ , then modulation localization, and apply Lemma 5.5 as before. Note that  $\widehat{w}_{3,m} = \mathcal{F}[\gamma(2^{[\alpha k_1]}(t - t_{k_4}) - m)v_{3,-}]$  is essentially supported on  $D_{k_3, \leq k_1^* + k_3 - 5}$ , and thus we can assume  $\max(j_1, j_2, j_4) \geq k_1^* + k_3 - 5 > j_3$ . Indeed, when  $j_3 > k_1^* + k_3 - 5$ , then it follows from the support consideration that

$$\begin{aligned} |\widehat{w}_{3,m}(n, \tau)| &\leq \int 2^{-[\alpha k_1]} |\widehat{\gamma}(2^{-[\alpha k_1]}(\tau - \tau'))| |\widehat{v}_{3,-}(n, \tau')| d\tau' \\ &\lesssim 2^{-10k_1^*} \int 2^{-[\alpha k_1]} |\phi(2^{-[\alpha k_1]}(\tau - \tau'))| |\widehat{v}_{3,-}(n, \tau')| d\tau' \end{aligned}$$

for some  $\phi \in \mathcal{S}$ . Then, we can simply apply Cauchy-Schwarz inequality to conclude this case.

The estimate (6.16) provides a good bound for  $s > -\frac{1}{8}$  when  $k_3$  is small in comparison to  $k_1$ , more precisely, when  $k_3 \leq \frac{2}{3}k_1$ . Otherwise, i.e. if  $k_3 > \frac{2}{3}k_1$ , we use the time localization

argument with the second claim in Lemma 5.5. Then, we obtain

$$\text{LHS of (6.14)} \lesssim 2^{(-\frac{1}{2}+\alpha)k_1^* - \frac{k_3}{2} + (\frac{1}{2}-\alpha)k_4 - \frac{\min(\alpha k_1^*, k_3)}{2}} \|u_1\|_{F_{k_1}^\alpha} \|u_2\|_{F_{k_2}^\alpha} \|u_3\|_{F_{k_3}^\alpha}.$$

This completes the proof of Lemma 6.6.  $\square$

**Remark 6.7.** When  $j_4 \geq [\alpha k_1]$ , we can use Lemma 3.7 (b) to improve the constant in (6.14) to  $2^{-(\frac{1}{2}-\varepsilon)k_1^* - \frac{\alpha}{2}k_4}$  when  $|k_3 - k_4| \geq 2$ , and  $2^{-(\frac{1}{2}-\varepsilon)k_1^* + \frac{k_3^*}{4} - \frac{\alpha}{2}k_4}$  when  $|k_3 - k_4| \leq 1$ . These estimates allow us to prove (6.1) for  $s > -\frac{1}{6}$  in the high modulation case:  $j_4 \geq [\alpha k_1]$ .

**Lemma 6.8** (low  $\times$  low  $\times$  low  $\rightarrow$  low). *If  $k_1, k_2, k_3, k_4 \leq 200$ , then we have*

$$\|\mathbf{P}_{k_4} \mathcal{N}(u_1, u_2, u_3)\|_{N_{k_4}^\alpha} + \|\mathbf{P}_{k_4} \mathcal{R}(u_1, u_2, u_3)\|_{N_{k_4}^\alpha} \lesssim \|u_1\|_{F_{k_1}^\alpha} \|u_2\|_{F_{k_2}^\alpha} \|u_3\|_{F_{k_3}^\alpha},$$

*Proof.* This follows immediately from Young's inequality on (6.10) with (3.6).  $\square$

**Summary:** We summarize the regularity restrictions from Lemmata 6.2 - 6.8. In the following, we assume  $s < 0$ .

(i) high  $\times$  low  $\times$  low  $\rightarrow$  high: In view of Lemma 6.2, we need to have

$$sk_4 - \frac{1}{2}(1-\varepsilon)k_1^* \leq s(k_1 + k_2 + k_3) \iff s(k_3^* + k_4^*) \geq -\frac{1}{2}(1-\varepsilon)k_1^*.$$

The latter holds for  $s \geq -\frac{1}{4}(1-\varepsilon)$ .

(ii) high  $\times$  high  $\times$  low  $\rightarrow$  high: In view of Lemma 6.3, we need to have

$$sk_4 - (1-\varepsilon)k_1^* \leq s(k_1 + k_2 + k_3) \iff s(k_1^* + k_4^*) \geq -(1-\varepsilon)k_1^*.$$

The latter holds for  $s \geq -\frac{1}{2}(1-\varepsilon)$ .

(iii) high  $\times$  high  $\times$  high  $\rightarrow$  high: In view of Lemma 6.4, we need to have  $-\frac{1}{2}(\alpha-\varepsilon) \leq 2s$ . Hence, it suffices to choose

$$\alpha = -4s + \varepsilon \tag{6.17}$$

for some sufficiently small  $\varepsilon = \varepsilon(s) > 0$ .

(iv) high  $\times$  high  $\times$  high  $\rightarrow$  low: In this case, we use Lemma 6.5. In particular, from (6.13) with  $M_1$ , we need to have

$$-(1-\alpha-\varepsilon) \leq 3s. \tag{6.18}$$

In view of (6.17), this holds for  $s \geq -\frac{1}{7} + \frac{2}{7}\varepsilon$ . Next, we consider the case  $s \leq -\frac{1}{7}$ . From (6.13) with  $M_2$ , we need to have

$$-(1 - \frac{3}{4}\alpha - \varepsilon)k_1^* + (s + \frac{1}{2} - \alpha)k_4^* \leq 3sk_1^*. \tag{6.19}$$

In view of (6.17), we have  $s \geq -\frac{1}{6} + \frac{7}{24}\varepsilon$  from the coefficients of  $k_1^*$ , while  $s \leq -\frac{1}{10} + \frac{\varepsilon}{5}$  guarantees that the coefficient of  $k_4^*$  is non-positive. Hence, (6.18) or (6.19) holds for  $s \in (-\frac{1}{6}, 0)$ .

(v) high  $\times$  high  $\times$  low  $\rightarrow$  low: First, we consider  $s > -\frac{1}{12}$ . In view of (6.14) with  $M_1$  of Lemma 6.6, we need to have

$$(-\frac{1}{2} + \alpha + \varepsilon)k_1^* \leq 2sk_1^* \quad \text{and} \quad (s - \alpha)k_4 - \beta \leq sk_3. \tag{6.20}$$

From (6.17), the first condition provides  $s \geq -\frac{1}{12} + \frac{1}{3}\varepsilon$ . The second condition is trivially satisfied when  $k_4 \geq k_3 - 5$ . When  $k_3 \geq k_4 + 5$ , it gives  $s \geq -\frac{1}{2}$ .

Next, we consider  $s \leq -\frac{1}{12}$ . First, we consider the case  $|k_3 - k_4| \leq 1$ . In view of (6.14) with  $M_2$  in Lemma 6.6, we need to have

$$(-\frac{1}{2} + \frac{1}{2}\alpha + \varepsilon)k_1^* \leq 2sk_1^* \quad \text{and} \quad (\frac{1}{4} - \alpha)k_3^* \leq 0. \quad (6.21)$$

From (6.17), the first condition provides  $s \geq -\frac{1}{8} + \frac{3}{8}\varepsilon$ , while the second condition provides  $s \leq -\frac{1}{16} + \frac{\varepsilon}{4}$ .

Next, let us consider the case when  $k_3 \geq k_4 + 2$ . (The case  $k_4 \geq k_3 + 2$  is easier. In this case, the condition yields  $s \in (-\frac{1}{8}, 0)$ .) In this case, from (6.14) with  $M_2$  in Lemma 6.6, we need to have

$$sk_4 + (-\frac{1}{2} + \frac{1}{2}\alpha + \varepsilon)k_1^* - \alpha k_4 \leq 2sk_1^* + sk_3. \quad (6.22)$$

When  $|n_3| \gg |n_4|$ , there is no gain of regularity on  $n_3$  from  $n_4$ . Namely, with (6.17) we have  $s \geq -\frac{1}{10} + \frac{3}{10}\varepsilon$  by collecting the coefficients of  $k_1^*$  and  $k_3$ . The coefficients of  $k_4$  give  $s \leq \frac{1}{5}\varepsilon$ , i.e. no restriction by taking  $\varepsilon > 0$  sufficiently small.

Lastly, we consider  $s \leq -\frac{1}{10}$ . Suppose that  $k_3 \leq \frac{2}{3}k_1$ . Then, from (6.15), we have

$$sk_4 - \frac{k_1^*}{2} + \frac{\alpha}{2}k_3 + (\frac{1}{2} - \alpha + \varepsilon)k_4 \leq 2sk_1^* + sk_3. \quad (6.23)$$

By writing  $-\frac{k_1^*}{2} \leq -(\frac{1}{4} - \varepsilon)k_1^* - \frac{3}{8}k_3 - \frac{3}{2}\varepsilon k_4$ , the coefficients of  $k_1^*$  and  $k_3$  yield  $s \geq -\frac{1}{8} + \frac{\varepsilon}{2}$  and  $s \geq -\frac{1}{8} + \frac{\varepsilon}{6}$ , respectively, while the coefficients of  $k_4$  yield  $s \leq -\frac{1}{10} + \frac{3}{10}\varepsilon$ . When  $k_3 > \frac{2}{3}k_1$ , the second term in (6.15) gives

$$sk_4 + (-\frac{1}{2} + \alpha)k_1^* - \frac{k_3}{2} + (\frac{1}{2} - \alpha)k_4 - \frac{1}{2}\min(\alpha k_1^*, k_3) \leq 2sk_1^* + sk_3. \quad (6.24)$$

In view of (6.19), we have  $\min(\alpha k_1^*, k_3) = \alpha k_1^*$  for  $s > -\frac{1}{8}$ . Thus, (6.24) yields the condition  $s \in (-\frac{1}{8}, -\frac{1}{40})$ .

Hence, (v.a) (6.20), or (v.b) (6.21) and one of (6.22), (6.23) or (6.24) holds for  $s \in (-\frac{1}{8}, 0)$ .

(vi) low  $\times$  low  $\times$  low  $\rightarrow$  low: There is no condition imposed in this case.

We conclude this section by presenting the proof of Proposition 6.1.

*Proof of Proposition 6.1.* First, we consider the first inequality (6.1). From the definitions, we have

$$\|\mathcal{N}(u_1, u_2, u_3)\|_{N^{s,\alpha}(T)}^2 = \sum_{k_4=0}^{\infty} 2^{2sk_4} \|\mathbf{P}_{k_4}\mathcal{N}(u_1, u_2, u_3)\|_{N_{k_4}^{\alpha}(T)}^2. \quad (6.25)$$

Applying a dyadic decomposition, we have

$$\|\mathbf{P}_{k_4}\mathcal{N}(u_1, u_2, u_3)\|_{N_{k_4}^{\alpha}(T)} \leq \sum_{k_1, k_2, k_3 \in \mathbb{Z}_+} \|\mathbf{P}_{k_4}\mathcal{N}(\mathbf{P}_{k_1}u_1, \mathbf{P}_{k_2}u_2, \mathbf{P}_{k_3}u_3)\|_{N_{k_4}^{\alpha}(T)}. \quad (6.26)$$

Given  $k \in \mathbb{Z}_+$ , let  $u_{k_i}$  be an extension of  $\mathbf{P}_{k_i}(u_i)$  such that  $\|u_{k_i}\|_{F_k^{\alpha}} \leq 2\|\mathbf{P}_{k_i}u_i\|_{F_k^{\alpha}(T)}$ . In view of (3.6), we can assume that  $u_{k_i}$  is supported on  $[-2T, 2T]$ . Then, we have

$$\|\mathbf{P}_{k_4}\mathcal{N}(\mathbf{P}_{k_1}u_1, \mathbf{P}_{k_2}u_2, \mathbf{P}_{k_3}u_3)\|_{N_{k_4}^{\alpha}(T)} \leq \sum_{k_1, k_2, k_3 \in \mathbb{Z}_+} \|\mathbf{P}_{k_4}\mathcal{N}(u_{k_1}, u_{k_2}, u_{k_3})\|_{N_{k_4}^{\alpha}}. \quad (6.27)$$

By Lemmata 6.2 - 6.8 and Summary (i) - (vi) along with Lemma 3.4, we obtain

$$\|\mathbf{P}_{k_4}\mathcal{N}(u_{k_1}, u_{k_2}, u_{k_3})\|_{N_{k_4}^{\alpha}} \lesssim T^{\theta} \prod_{i=1}^3 2^{(s-\varepsilon)k_i} \|u_{k_i}\|_{F_{k_i}^{\alpha}} \quad (6.28)$$

for some small  $\theta, \varepsilon > 0$ . Here, in order to obtain a small power of  $T$ , we actually applied Lemmata 6.2 - 6.8 with the right-hand sides with  $F_{k_i}^{b,\alpha}$  (see Lemma 3.4) for some  $b = \frac{1}{2} -$  at a slight loss of the regularity in the highest modulation. Putting (6.25) - (6.28) together, we obtain (6.1). The second estimate (6.2) follows from a similar argument with Lemmata 6.4 and 6.8.  $\square$

## 7. ENERGY ESTIMATES ON SMOOTH SOLUTIONS

In this section, we establish an energy estimate for (smooth) solutions to the Wick ordered cubic NLS (1.3). This argument is very close in spirit to the  $I$ -method developed by Colliander-Keel-Staffilani-Takaoka-Tao [14, 15, 16]. Let  $u \in C(\mathbb{R}; H^\infty(\mathbb{T}))$  be a smooth solution to (1.3). Then, by Fundamental Theorem of Calculus, we have

$$\begin{aligned}
\|u(t)\|_{H^s}^2 - \|u(0)\|_{H^s}^2 &= \|a(t)\|_{H^s}^2 - \|a(0)\|_{H^s}^2 = 2\operatorname{Re} \left( \int_0^t \sum_n \langle n \rangle^{2s} \partial_t a_n(t') \bar{a}_n(t') dt' \right) \\
&= 2\operatorname{Re} i \left( \int_0^t \sum_n \langle n \rangle^{2s} \mathbf{N}(a)_n \bar{a}_n(t') dt' \right) \\
&\quad - 2\operatorname{Re} i \left( \int_0^t \sum_n \langle n \rangle^{2s} \mathbf{R}(a)_n \bar{a}_n(t') dt' \right) \\
&= 2\operatorname{Re} i \left( \int_0^t \sum_n \langle n \rangle^{2s} \sum_{\substack{n=n_1-n_2+n_3 \\ n_2 \neq n_1, n_3}} e^{-i\Phi(\bar{n})t'} a_{n_1} \bar{a}_{n_2} a_{n_3} \bar{a}_n(t') dt' \right) \\
&\quad - 2\operatorname{Re} i \left( \int_0^t \sum_n \langle n \rangle^{2s} |a_n(t')|^4 dt' \right), \tag{7.1}
\end{aligned}$$

where  $\mathbf{N}(a)$  and  $\mathbf{R}(a)$  are as in (2.4) and  $\Phi(\bar{n})$  is as in (2.5). Here,  $a_n(t) = e^{-itn^2} \hat{u}_n(t)$  denotes (the Fourier coefficient of) the interaction representation of  $u$  defined in Section 2. Clearly, the second term on the right-hand side of (7.1) is 0. Moreover, letting  $n_4 = n$  and symmetrizing under the summation indices  $n_1, \dots, n_4$ , we obtain

$$\begin{aligned}
\|u(t)\|_{H^s}^2 - \|u(0)\|_{H^s}^2 &= -\frac{i}{2} \int_0^t \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3}} \Psi_s(\bar{n}) e^{-i\Phi(\bar{n})t'} a_{n_1} \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4}(t') dt' \\
&=: R_4(t), \tag{7.2}
\end{aligned}$$

where  $\Psi_s(\bar{n})$  is defined by

$$\Psi_s(\bar{n}) = \langle n_1 \rangle^{2s} - \langle n_2 \rangle^{2s} + \langle n_3 \rangle^{2s} - \langle n_4 \rangle^{2s} \tag{7.3}$$

and  $\Phi(\bar{n})$  is as in (2.5) with  $n$  replaced by  $n_4$ .

**Remark 7.1.** It is this symmetrization process that fails when we try to establish an energy estimate for a difference of two solutions. The symbol  $\Psi_s(\bar{n})$  provides an extra decay via Mean Value Theorem and Double Mean Value Theorem (Lemmata 4.1 and 4.2 in [16].) See (7.11), (7.14), and (7.15). This is crucial in estimating  $R_4(t)$  in the *nearly resonant* case (i.e.  $|\Phi(\bar{n})| \ll n_1^*$ ) below  $L^2(\mathbb{T})$ .

With the  $L^6$ -Strichartz estimate (5.3) and its refinement (Lemma 7.2), we can estimate  $R_4$  in (7.2) and obtain an energy estimate for  $s > -\frac{1}{10}$ . However, for  $s \leq -\frac{1}{10}$ , we need to add a “correction term” to (7.2) as in the application of the  $I$ -method [15, 16]. In terms



of the interaction representation  $a_n(t)$ , this process can be regarded as the Poincaré-Dulac normal form reduction applied to the evolution equation satisfied by  $\|a(t)\|_{H^s}^2$ . In fact, integrating (7.2) by parts, we have

$$\begin{aligned} \|u(t)\|_{H^s}^2 - \|u(0)\|_{H^s}^2 &= \frac{1}{2} \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} e^{-i\Phi(\bar{n})t'} a_{n_1} \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4}(t') \Big|_0^t \\ &\quad - \frac{1}{2} \int_0^t \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} e^{-i\Phi(\bar{n})t'} \partial_t (a_{n_1} \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4})(t') dt' \\ &=: \Lambda_4(t; u) - \Lambda_4(0; u) + R_6(t; u), \end{aligned}$$

where the correction term  $\Lambda_4(t; u)$  is given by

$$\Lambda_4(t) = \Lambda_4(t; u) = \frac{1}{2} \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} \hat{u}_{n_1} \bar{\hat{u}}_{n_2} \hat{u}_{n_3} \bar{\hat{u}}_{n_4}(t).$$

For simplicity, we assume that the time derivative in  $R_6(t)$  falls on the first factor. The same comment applies to  $R_6^M(t)$  defined in (7.5). Using the equation (2.4), we can then write  $R_6(t) := R_6(t; u)$  as

$$\begin{aligned} R_6(t) &= c \int_0^t \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} e^{-i\Phi(\bar{n})t'} \\ &\quad \times \sum_{\substack{n_1=n_5-n_6+n_7 \\ n_6 \neq n_5, n_7}} e^{-i\tilde{\Phi}(\bar{n})t'} a_{n_5} \bar{a}_{n_6} a_{n_7} \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4}(t') dt' \\ &\quad + c \int_0^t \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} e^{-i\Phi(\bar{n})t'} |a_{n_1}|^2 a_{n_1} \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4}(t') dt' \\ &= c \int_0^t \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} \sum_{\substack{n_1=n_5-n_6+n_7 \\ n_6 \neq n_5, n_7}} \hat{u}_{n_5} \bar{\hat{u}}_{n_6} \hat{u}_{n_7} \bar{\hat{u}}_{n_2} \hat{u}_{n_3} \bar{\hat{u}}_{n_4}(t') dt' \\ &\quad + c \int_0^t \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} |\hat{u}_{n_1}|^2 \hat{u}_{n_1} \bar{\hat{u}}_{n_2} \hat{u}_{n_3} \bar{\hat{u}}_{n_4}(t') dt', \end{aligned}$$

where the phase function  $\tilde{\Phi}(\bar{n})$  is given by

$$\begin{aligned} \tilde{\Phi}(\bar{n}) &= \Phi(n_5, n_6, n_7, n_1) = n_1^2 - n_5^2 + n_6^2 - n_7^2 \\ &= 2(n_6 - n_5)(n_6 - n_7) = 2(n_1 - n_5)(n_1 - n_7). \end{aligned} \tag{7.4}$$

The boundary term  $\Lambda_4$  corresponds to the correction term in an application of the  $I$ -method. While  $\Lambda_4$  satisfies a good estimate in terms of the regularity, it does not see the length of the time interval. In order to gain a small power of  $T > 0$ , we need to modify the argument above.

Let  $R_4^M(t)$  be the part of  $R_4(t)$ , where all the frequencies  $|n_j| \leq M$ . Namely, we have

$$\begin{aligned} R_4^M(t) &= -\frac{i}{2} \int_0^t \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3 \\ |n_j| \leq M}} \Psi_s(\bar{n}) e^{-i\Phi(\bar{n})t'} a_{n_1} \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4}(t') dt' \\ &= -\frac{i}{2} \int_0^t \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3 \\ |n_j| \leq M}} \Psi_s(\bar{n}) \widehat{u}_{n_1} \bar{\widehat{u}}_{n_2} \widehat{u}_{n_3} \bar{\widehat{u}}_{n_4}(t') dt'. \end{aligned}$$

Then, by applying a normal form reduction to  $R_4(t) - R_4^M(t)$ , we have

$$\|u(t)\|_{H^s}^2 - \|u(0)\|_{H^s}^2 = R_4^M(t; u) + \Lambda_4^M(t; u) - \Lambda_4^M(0; u) + R_6^M(t; u),$$

where  $\Lambda_4^M(t) := \Lambda_4^M(t; u)$  and  $R_6^M(t) := R_6^M(t; u)$  are given by

$$\Lambda_4^M(t) = \frac{1}{2} \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3 \\ \max |n_j| > M}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} \widehat{u}_{n_1} \bar{\widehat{u}}_{n_2} \widehat{u}_{n_3} \bar{\widehat{u}}_{n_4}(t)$$

and

$$\begin{aligned} R_6^M(t) &= c \int_0^t \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3 \\ \max |n_j| > M}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} \sum_{\substack{n_1=n_5-n_6+n_7 \\ n_6 \neq n_5, n_7}} \widehat{u}_{n_5} \bar{\widehat{u}}_{n_6} \widehat{u}_{n_7} \bar{\widehat{u}}_{n_2} \widehat{u}_{n_3} \bar{\widehat{u}}_{n_4}(t') dt' \\ &\quad + c \int_0^t \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_2 \neq n_1, n_3 \\ \max |n_j| > M}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} |\widehat{u}_{n_1}|^2 \widehat{u}_{n_1} \bar{\widehat{u}}_{n_2} \widehat{u}_{n_3} \bar{\widehat{u}}_{n_4}(t') dt' =: \text{I}(t) + \text{II}(t). \end{aligned} \quad (7.5)$$

In the remaining part of this section, we establish multilinear estimates on  $R_4^M$ ,  $\Lambda_4^M$ , and  $R_6^M$ . In the following,  $u$  denotes a smooth solution to (1.3). The main tool is the following refinement of the  $L^6$ -Strichartz estimate.

**Lemma 7.2.** *Let  $f_i$  be supported in  $D_{k_i, \leq j_i}$ ,  $i = 1, 2, 3$ . Then, the following estimate holds:*

$$\|\mathbf{1}_{I_{k_4}}(n) \cdot f_1 * \widetilde{f}_2 * f_3\|_{\ell_n^2 L_\tau^2} \lesssim M \prod_{i=1}^3 2^{\frac{j_i}{2}} \|f_i\|_{\ell_n^2 L_\tau^2},$$

where  $\widetilde{f}_2(n_2, \tau_2) = f_2(-n_2, -\tau_2)$ . Here,  $M$  is given as follows.

(a) If  $|k_1 - k_4| \geq 10$  and  $\max(k_1, k_4) \geq k_1^* - 5$ , then we have

$$M = \min(2^{\frac{k_1}{2}}, 2^{\frac{k_4}{2}}) \min_{i=2,3} \left\{ (1 + 2^{j_i - k_1^*})^{\frac{1}{2}} 2^{\frac{-j_i}{2}} \right\}. \quad (7.6)$$

(b) If  $k_2 \leq k_4 - 10 \leq k_1^* - 20$  or  $|n_1 - n_3| \sim 2^{k_1^*}$ , then we have

$$M = 2^{\frac{k_2}{2}} \min_{i=1,3} \left\{ (1 + 2^{j_i - k_1^*})^{\frac{1}{2}} 2^{\frac{-j_i}{2}} \right\}. \quad (7.7)$$

*Proof.* The first bound (7.6) follows from duality and (a slight modification of) Cases (a) and (b) in the proof of Lemma 5.3. The second bound (7.7) follows from (5.11) and (5.12)

in the proof of Lemma 5.5. In either case, we have  $|\partial_{n_3} h_1(n_3)| = |2(n_2 - 2n_3 + n_4)| = |2(n_1 - n_3)| \sim 2^{k_1^*}$ , where  $h_1$  is as in (5.12). Thus, we have

$$\begin{aligned}
(5.11) &\lesssim (1 + 2^{j_1 - k_1^*})^{\frac{1}{2}} 2^{-\frac{j_1}{2}} \int \sum_{n_4} |g_4(n_4, \tau_4)| \sum_{n_2} |g_2(n_2, \tau_2)| \left( \sum_{n_3} |g_3(n_3, \tau_3)|^2 \right. \\
&\quad \times \left. |g_1(n_2 - n_3 + n_4, h_1(n_2, n_3, n_4, \tau_2, \tau_3, \tau_4))|^2 \right)^{\frac{1}{2}} d\tau_2 d\tau_3 d\tau_4 \\
&\lesssim (1 + 2^{j_1 - k_1^*})^{\frac{1}{2}} 2^{-\frac{j_1}{2}} \cdot 2^{\frac{k_2}{2}} \|g_4(n_4, \tau_4)\|_{\ell_{n_4}^2 L_{\tau_4}^2} \int \|g_2(n_2, \tau_2)\|_{\ell_{n_2}^2} \sup_{n_2} \left( \sum_{n_3, n_4} |g_3(n_3, \tau_3)|^2 \right. \\
&\quad \times \left. \|g_1(n_2 - n_3 + n_4, h_1(n_2, n_3, n_4, \tau_2, \tau_3, \tau_4))\|_{L_{\tau_4}^2}^2 \right)^{\frac{1}{2}} d\tau_2 d\tau_3 \\
&\lesssim (1 + 2^{j_1 - k_1^*})^{\frac{1}{2}} 2^{-\frac{j_1}{2}} \cdot 2^{\frac{k_2}{2}} \|\mathcal{F}(u_4)\|_{\ell_n^2 L_\tau^2} \prod_{i=1}^3 2^{\frac{j_i}{2}} \|\mathcal{F}(u_i)\|_{\ell_n^2 L_\tau^2}.
\end{aligned}$$

Then, the second bound (7.7) follows from duality.  $\square$

**Proposition 7.3.** *Let  $s \in (-\frac{1}{4}, 0)$  and  $\alpha = -4s + \varepsilon$  as in (6.17). Then, there exists  $\theta > 0$  such that*

$$|R_4^M(T; u)| \lesssim T^\theta M^{c(s)} \|u\|_{F^{s, \alpha}(T)}^4 \quad (7.8)$$

for  $T \in (0, 1]$ , where  $c(s)$  is defined by

$$c(s) = \max(-\frac{1}{2} - 5s, 0). \quad (7.9)$$

**Remark 7.4.** It follows from (7.9) that we have  $c(s) = 0$  for  $s > -\frac{1}{10}$ . Namely, for  $s > -\frac{1}{10}$ , we do not need to add a correction term. In this case, (7.2) and Proposition 7.3 (with  $M = \infty$ ) yield a good energy estimate:

$$|\|u\|_{E^s(T)}^2 - \|u(0)\|_{H^s}^2| \leq |R_4(T)| \lesssim T^\theta \|u\|_{F^{s, \alpha}(T)}^4,$$

allowing us to prove the existence result (Theorem 1.2) simply by Proposition 7.3 along with the argument in Section 8. This is basically the periodic analogue of the result in [13].

When  $s \leq -\frac{1}{10}$ , we have  $c(s) > 0$  (see (7.19) below) and hence we need to add the correction term  $\Lambda_4^M$ .

*Proof.* Apply a dyadic decomposition on the spatial frequencies  $|n_i| \sim 2^{k_i}$ ,  $i = 1, \dots, 4$ . By symmetry, assume that  $|n_1| \sim n_1^* := \max(|n_1|, |n_2|, |n_3|, |n_4|) \leq M$ . By assuming that each factor  $u_i$  has its Fourier support on  $I_{k_i} \times \mathbb{R}$ , it suffices to prove

$$|R_4^M(T)| \lesssim T^\theta M^{c(s)} \prod_{i=1}^4 2^{(s-)^{k_i}} \|\mathbf{P}_{k_i} u\|_{F_{k_i}^\alpha(T)}.$$

Here, a slight extra decay is needed to sum over dyadic blocks. Let  $\tilde{u}_i$  be an extension of  $u_i$  such that  $\|\tilde{u}_i\|_{F_{k_i}^\alpha} \leq 2\|\mathbf{P}_{k_i} u\|_{F_{k_i}^\alpha(T)}$ . For notational simplicity, we denote  $\tilde{u}_i$  by  $u_i$ ,  $i = 1, \dots, 4$ , in the following. Letting  $\gamma : \mathbb{R} \rightarrow [0, 1]$  be a smooth cutoff function supported

on  $[-1, 1]$  such that  $\sum_{m \in \mathbb{Z}} \gamma^4(t - m) \equiv 1$  for all  $t \in \mathbb{R}$ , we have

$$R_4^M(T) = \int_{\mathbb{R}} \mathbf{1}_{[0, T]}(t) \sum_{|m| \leq T2^{[\alpha K]}} \gamma^4(2^{[\alpha K]}t - m) \\ \times \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ n_2 \neq n_1, n_3 \\ |n_j| \leq M}} \Psi_s(\bar{n}) \widehat{u}_1(n_1) \overline{\widehat{u}_2(n_2)} \widehat{u}_3(n_3) \overline{\widehat{u}_4(n_4)}(t) dt.$$

Here,  $K = k_1^* + c$ . With  $f_{i,m}(t) = \gamma(2^{[\alpha K]}t - m)u_i(t)$ , let  $f_{i,j_i,m} = \mathcal{F}^{-1}[\eta_{j_i}(\tau - n^2)\widehat{f}_{i,m}]$ ,  $i = 1, \dots, 4$ . Then, it suffices to prove

$$\left| \int_{\mathbb{R}} \mathbf{1}_{[0, T]}(t) \sum_{j_1, \dots, j_4} \sum_{|m| \leq T2^{[\alpha K]}} \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ n_2 \neq n_1, n_3 \\ |n_j| \leq M}} \Psi_s(\bar{n}) \widehat{f}_{1,j_1,m}(n_1) \overline{\widehat{f}_{2,j_2,m}(n_2)} \right. \\ \left. \times \widehat{f}_{3,j_3,m}(n_3) \overline{\widehat{f}_{4,j_4,m}(n_4)}(t) dt \right| \lesssim T^\theta M^{c(s)} \prod_{i=1}^4 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha}. \quad (7.10)$$

In view of (3.5), we assume that  $j_i \geq \alpha K$ ,  $i = 1, \dots, 4$ . In the following, we prove (7.10) for fixed  $j_i$ ,  $i = 1, \dots, 4$ . For simplicity, we denote  $f_{i,j_i,m}$  by  $f_{i,j_i}$ . Lastly, note that from (2.5) with  $\tau_1 - \tau_2 + \tau_3 - \tau_4 = 0$ , we have

$$\sigma_1^* := \max(|\sigma_1|, |\sigma_2|, |\sigma_3|, |\sigma_4|) \gtrsim |\Phi(\bar{n})|,$$

where  $\sigma_j = \tau_j - n_j^2$ .

Define the subsets  $\mathcal{A}$  and  $\mathcal{B}$  of  $\{m \in \mathbb{Z} : |m| \leq T2^{[\alpha K]}\}$  by

$$\mathcal{A} = \{m \in \mathbb{Z} : \mathbf{1}_{[0, T]}(t) \gamma(2^{[\alpha K]}t - m) = \gamma(2^{[\alpha K]}t - m)\},$$

$$\mathcal{B} = \{m \in \mathbb{Z} : \mathbf{1}_{[0, T]}(t) \gamma(2^{[\alpha K]}t - m) \neq \gamma(2^{[\alpha K]}t - m) \text{ and } \mathbf{1}_{[0, T]}(t) \gamma(2^{[\alpha K]}t - m) \not\equiv 0\}.$$

**Part 1:** First, we consider the terms with  $m \in \mathcal{A}$ . In this case, we can drop the sharp cut-off  $\mathbf{1}_{[0, T]}(t)$  on the left-hand side of (7.10). We prove (7.10) with  $\theta = 1$  in this case.

• **Case (a):**  $|n_4 - n_1|, |n_4 - n_3| \ll n_1^*$ . (This case includes the nearly resonant case:  $|\Phi(\bar{n})| \ll n_1^*$ .)

Since  $n_1 - n_2 + n_3 - n_4 = 0$ , it follows that  $|n_1| \sim |n_2| \sim |n_3| \sim |n_4| \sim n_1^*$  in this case. Then, by Double Mean Value Theorem [16, Lemma 4.2], we have

$$|\Psi_s(\bar{n})| \lesssim (n_1^*)^{2s-2} |(n_4 - n_1)(n_4 - n_3)| \sim (n_1^*)^{2s-2} |\Phi(\bar{n})|. \quad (7.11)$$

By crudely estimating  $|\Phi(\bar{n})| \lesssim (n_1^*)^2$ , it follows from (7.11) with (6.17) that

$$\sum_{|m| \leq T2^{[\alpha K]}} (\sigma_1^*)^{-\frac{1}{2}} |\Psi_s(\bar{n})| \lesssim T(n_1^*)^{-1-2s+} \leq TM^{c(s)} (n_1^*)^{4s-}, \quad (7.12)$$

where

$$c(s) = \max(-1 - 6s+, 0). \quad (7.13)$$

Then, by Lemma 5.2 with (7.12), we obtain

$$|R_4^M(T)| \lesssim TM^{c(s)} \prod_{i=1}^4 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha}.$$

- **Case (b):**  $|n_4 - n_1| \sim n_1^* \gg |n_4 - n_3|$ .

In this case, we have  $|n_2| \sim |n_1| \sim n_1^*$ . Then, by Mean Value Theorem, we have

$$|\langle n_1 \rangle^{2s} - \langle n_2 \rangle^{2s}| \lesssim (n_1^*)^{2s-1} |n_1 - n_2| = (n_1^*)^{2s-1} |n_4 - n_3|. \quad (7.14)$$

- Subcase (b.i):  $|n_4 - n_3| \ll n_3^*$ .

In this case, we have  $|n_3| \sim |n_4|$ . Then, by Mean Value Theorem, we have

$$|\langle n_3 \rangle^{2s} - \langle n_4 \rangle^{2s}| \lesssim (n_3^*)^{2s-1} |n_4 - n_3|. \quad (7.15)$$

From (7.14) and (7.15), we have

$$|\Psi_s(\bar{n})| \lesssim (n_3^*)^{2s-1} |n_4 - n_3|. \quad (7.16)$$

Then, with (6.17) and (7.16), we have

$$\begin{aligned} \sum_{|m| \leq T2^{[\alpha K]}} (\sigma_1^*)^{-\frac{1}{2}} |\Psi_s(\bar{n})| &\lesssim T(n_1^*)^{-\frac{1}{2}-4s+} (n_3^*)^{2s-1} |n_4 - n_3|^{\frac{1}{2}} \lesssim T(n_1^*)^{-\frac{1}{2}-4s+} (n_3^*)^{2s-\frac{1}{2}} \\ &\lesssim T(n_1^*)^{-\frac{1}{2}-6s+} (n_3^*)^{-\frac{1}{2}+} \left( \prod_{i=1}^4 2^{(s-)^{k_i}} \right). \end{aligned} \quad (7.17)$$

If  $n_3^* \sim n_1^*$ , then we have (7.17)  $\lesssim TM^{c(s)}(n_1^*)^{4s-}$ , where  $c(s)$  is as in (7.13). Then, the rest follows as in Case (a).

Now, suppose  $n_3^* \ll n_1^*$ . We only consider the case  $\sigma_1 = \sigma_1^*$ . (A similar argument holds for other cases.) By Lemma 7.2 (a) and the time localization of size  $\sim 2^{k_1^*}$ , we have

$$\sum_{j_2, \dots, j_4} \|\overline{f_{2,j_2}} f_{3,j_3} \overline{f_{4,j_4}}\|_{L_{x,t}^2} \lesssim (n_1^*)^{-\frac{\alpha}{2}} (n_3^*)^{\frac{1}{2}} \prod_{i=2}^4 \|u_i\|_{F_{k_i}^\alpha}. \quad (7.18)$$

Here, we used the fact  $\alpha = -4s + \varepsilon \leq 1$  for  $s > -\frac{1}{4}$ . From (7.17) and (7.18) we obtain

$$\begin{aligned} |R_4^M(T)| &\lesssim \sum_{|m| \leq T2^{[\alpha K]}} \sum_{j_1, \dots, j_4} (\sigma_1^*)^{-\frac{1}{2}} |\Psi_s(\bar{n})| \|\mathcal{F}^{-1}(\sigma_1^{\frac{1}{2}} \widehat{f_{1,j_1}})\|_{L_{x,t}^2} \|\overline{f_{2,j_2}} f_{3,j_3} \overline{f_{4,j_4}}\|_{L_{x,t}^2} \\ &\lesssim TM^{c(s)} \prod_{i=1}^4 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha}, \end{aligned}$$

where

$$c(s) = \max(-\frac{1}{2} - 4s +, 0).$$

- Subcase (b.ii):  $|n_4 - n_3| \sim n_3^*$ .

In this case, we have  $|\Phi(\bar{n})| \gtrsim n_1^* n_3^*$ . Then, with (6.17), we have

$$\begin{aligned} \sum_{|m| \leq T2^{[\alpha K]}} (\sigma_1^*)^{-\frac{1}{2}} |\Psi_s(\bar{n})| &\lesssim T(n_1^*)^{-\frac{1}{2}-4s+} (n_3^*)^{-\frac{1}{2}} (n_4^*)^{2s} \\ &\lesssim T(n_1^*)^{-\frac{1}{2}-6s+} (n_3^*)^{-\frac{1}{2}-s} (n_4^*)^s \left( \prod_{i=1}^4 2^{(s-)^{k_i}} \right). \end{aligned}$$

Note that we have  $n_3^* \ll n_1^*$  in this case. Then, the rest follows from Lemma 7.2 (a) as in Subcase (b.i), where  $c(s)$  is given by

$$c(s) = \max(-\frac{1}{2} - 5s +, 0). \quad (7.19)$$

- **Case (c):**  $|n_4 - n_1|, |n_4 - n_3| \sim n_1^*$ .

In this case, we have  $|\Phi(\bar{n})| \sim (n_1^*)^2$ . Then, with (6.17), we have

$$\sum_{|m| \leq T2^{[\alpha K]}} (\sigma_1^*)^{-\frac{1}{2}} |\Psi_s(\bar{n})| \lesssim T(n_1^*)^{-1-4s+} (n_4^*)^{2s} \lesssim T(n_1^*)^{-1-7s+} (n_4^*)^s \left( \prod_{i=1}^4 2^{(s-)^{k_i}} \right).$$

Then, the rest follows from Lemma 5.2 as in Case (a) with  $c(s)$  given by

$$c(s) = \max(-1 - 7s, 0).$$

**Part 2:** Next, we consider the terms with  $m \in \mathcal{B}$ . In this case, we need to handle the sharp cutoff  $\mathbf{1}_{[0,T]}$ . The modification is systematic and thus we only discuss Case (a) when  $\sigma_1 = \sigma_1^*$ . The main point is that we do not need to sum over  $m$  since there are only  $O(1)$  many values of  $m$  in  $\mathcal{B}$ . In particular, we gain  $(n_1^*)^{-\alpha}$  as compared to Cases (a), (b), and (c).

From (7.11), we have

$$(\sigma_1^*)^{-\frac{1}{2}+2\theta+} |\Psi(\bar{n})| \lesssim (n_1^*)^{-1-2s} \leq M^{c(s)} (n_1^*)^{4s-}$$

for  $\theta < -s$ , where  $c(s)$  is as in (7.13). Compare this with (7.12). Proceeding as in Case (a), we obtain

$$\begin{aligned} |R_4^M(T)| &\lesssim M^{c(s)} (n_1^*)^{4s-} \left( \sum_{j_1} 2^{-\theta j_1} \right) \sup_{j_1} 2^{(\frac{1}{2}-\theta-)j_1} \|\mathcal{F}(\mathbf{1}_{[0,T]} f_{1,j_1})\|_{\ell_n^2 L_\tau^2} \sum_{j_2, \dots, j_4} \prod_{i=2}^4 \|f_{i,j_i}\|_{L_{x,t}^6} \\ &\lesssim T^\theta M^{c(s)} \prod_{i=1}^4 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha}, \end{aligned}$$

where we used Lemmata 3.5 and 3.4 in the last step. In all the other cases, we can save a small power of  $\sigma_1^*$  thanks to the gain of  $(n_1^*)^{-\alpha}$ . Since the modifications are similar, we omit details. This completes the proof of Proposition 7.3.  $\square$

Next, we estimate the correction term  $\Lambda_4^M$ .

**Proposition 7.5.** *Let  $s \in (-\frac{1}{2}, 0)$  and  $\alpha > 0$ . Then, there exists  $d(s) > 0$  such that*

$$|\Lambda_4^M(t; u)| \lesssim M^{-d(s)} \|u\|_{F^{s,\alpha}(T)}^4 \quad (7.20)$$

for  $t \in [0, T] \subset [0, 1]$ .

*Proof.* After applying a dyadic decomposition on the spatial frequencies, assume that  $|n_1| \sim n_1^* > M$ . In view of Lemma 3.3 and the fact that  $\max(|n_j|) > M$ , it suffices to prove

$$|\Lambda_4^M(t; u)| \lesssim \|u(t)\|_{H^{s-}}^4 \quad (7.21)$$

for  $s \in (-\frac{1}{2}, 0)$ .

- **Case (a):**  $|n_4 - n_1|, |n_4 - n_3| \ll n_1^*$ .

From (7.11), we have

$$\left| \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} \right| \lesssim (n_1^*)^{2s-2} \lesssim \prod_{i=1}^4 \langle n_i \rangle^{\frac{s}{2}-\frac{1}{2}}. \quad (7.22)$$

Then, by Hölder (in  $x$ ) and Sobolev inequalities, we have

$$|\Lambda_4^M(t; u)| \lesssim \|u(t)\|_{W^{\frac{s}{2}-\frac{1}{2},4}}^4 \lesssim \|u(t)\|_{H^{\frac{s}{2}-\frac{1}{4}}}^4.$$

Hence, (7.21) follows once we note that  $\frac{s}{2} - \frac{1}{4} < s$  for  $s > -\frac{1}{2}$ .

• **Case (b):**  $|n_4 - n_1| \sim n_1^* \gg |n_4 - n_3|$ .

◦ Subcase (b.i): We first consider the case  $|n_4 - n_3| \ll n_3^*$ . From (7.16), we have

$$\left| \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} \right| \lesssim (n_1^*)^{-1} (n_3^*)^{2s-1} \lesssim (n_1^*)^{-1} (n_3^*)^{s-\frac{1}{2}} (n_4^*)^{s-\frac{1}{2}}. \quad (7.23)$$

Then, by Hölder (in  $x$ ) and Sobolev inequalities (note that  $\frac{4}{1-2s} \geq 2$  for  $s \geq -\frac{1}{2}$ ), we have

$$|\Lambda_4(t; u)| \lesssim \|u\|_{W^{-\frac{1}{2}, \frac{4}{1-2s}}}^2 \|u\|_{W^{s-\frac{1}{2}, \frac{4}{1+2s}}}^2 \lesssim \|u(t)\|_{H^{\frac{s}{2}-\frac{1}{4}}}^4.$$

Hence, (7.21) once we note that  $\frac{s}{2} - \frac{1}{4} < s$  for  $s > -\frac{1}{2}$ .

◦ Subcase (b.ii): When  $|n_4 - n_3| \sim n_3^*$ , we have  $|\Phi(\bar{n})| \gtrsim n_1^* n_3^*$ . Thus, we have

$$\left| \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} \right| \lesssim (n_1^*)^{-1} (n_3^*)^{-1} (n_4^*)^{2s} \lesssim (n_1^*)^{-1} (n_3^*)^{s-\frac{1}{2}} (n_4^*)^{s-\frac{1}{2}}. \quad (7.24)$$

The rest follows as in Subcase (b.i).

• **Case (c):**  $|n_4 - n_1|, |n_4 - n_3| \sim n_1^*$ .

In this case, we have  $|\Phi(\bar{n})| \sim (n_1^*)^2$  and thus

$$\left| \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} \right| \lesssim (n_1^*)^{-2} (n_4^*)^{2s}. \quad (7.25)$$

Suppose  $n_4^* = |n_4|$ . Then, by Hölder (in  $x$ ) and Sobolev inequalities, we have

$$|\Lambda_4(t; u)| \lesssim \|u\|_{W^{-\frac{2}{3}, \frac{6}{1-2s}+}}^3 \|u\|_{W^{2s, \frac{2}{1+2s}-}} \lesssim \|u\|_{H^{\frac{s}{3}-\frac{1}{3}+}}^3 \|u\|_{H^{s-}}.$$

Hence, (7.21) once we note that  $\frac{s}{3} - \frac{1}{3} < s$  for  $s > -\frac{1}{2}$ . □

Finally, we estimate  $R_6^M$  in (7.5).

**Proposition 7.6.** *Let  $s \in (-\frac{1}{8}, 0)$  and  $\alpha = -4s + \varepsilon$  as in (6.17). Then, there exists  $\theta > 0$  such that*

$$|R_6^M(T; u)| \lesssim T^\theta \|u\|_{F^{s, \alpha}(T)}^6 \quad (7.26)$$

for  $T \in (0, 1]$ .

*Proof.* First, we estimate the contribution from  $\Pi(T)$ . Since  $n_1 - n_2 + n_3 - n_4 = 0$ , we have  $|n_j| \gtrsim |n_1|$  for some  $j \in \{2, 3, 4\}$ , say  $|n_2| \gtrsim |n_1|$ . Then, (7.26) follows (with  $\theta = 1$ ) from Proposition 7.5 and Lemma 3.3 by noting the following. In Cases (a) and (b) of Proposition 7.5, we have

$$\langle n_1 \rangle^{\frac{s}{2}-\frac{1}{4}+} \langle n_2 \rangle^{\frac{s}{2}-\frac{1}{4}+} \lesssim \langle n_1 \rangle^{3s} \langle n_2 \rangle^s$$

as long as  $s > -\frac{1}{6}$ . In Case (c), by writing  $(n_1^*)^{-2} = (n_1^*)^{-2-2s} (n_1^*)^{2s}$ , it suffices to note that  $-\frac{s}{3} - \frac{1}{3} + < s$  for  $s > -\frac{1}{4}$ .

Next, we consider the contribution from  $\mathbf{I}(T)$ . Apply a dyadic decomposition on the spatial frequencies  $|n_i| \sim 2^{k_i}$ ,  $i = 1, \dots, 7$ . For convenience, let  $n_1^* \geq n_2^* \geq n_3^* \geq n_4^*$ ,  $m_1^* \geq m_2^* \geq m_3^*$  be the decreasing rearrangements of  $\{|n_1|, |n_2|, |n_3|, |n_4|\}$ , and  $\{|n_5|, |n_6|, |n_7|\}$ , respectively. Then, by assuming that each factor  $u_i$  has its Fourier support on  $I_{k_i} \times \mathbb{R}$ , it suffices to prove

$$|\mathbf{I}(T)| \lesssim T^\theta \prod_{i=2}^7 2^{(s-)^{k_i}} \|\mathbf{P}_{k_i} u\|_{F_{k_i}^\alpha(T)}$$

for  $s > -\frac{1}{8}$ . Here, a slight extra decay is needed to sum over dyadic blocks. Let  $\tilde{u}_i$  be an extension of  $u_i$  such that  $\|\tilde{u}_i\|_{F_{k_i}^\alpha} \leq 2\|\mathbf{P}_{k_i} u\|_{F_{k_i}^\alpha(T)}$ . For notational simplicity, we denote  $\tilde{u}_i$  by  $u_i$ ,  $i = 2, \dots, 7$ , in the following. Letting  $\gamma : \mathbb{R} \rightarrow [0, 1]$  be a smooth cutoff function supported on  $[-1, 1]$  such that  $\sum_m \gamma^6(t - m) \equiv 1$  for all  $t \in \mathbb{R}$ , we have

$$\begin{aligned} \mathbf{I}(T) &= \int_{\mathbb{R}} \mathbf{1}_{[0, T]}(t) \sum_{|m| \leq T2^{[\alpha K]}} \gamma^6(2^{[\alpha K]}t - m) \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ n_2 \neq n_1, n_3 \\ \max(|n_j|) > M}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} \mathbf{1}_{I_{k_1}}(n_1) \\ &\quad \times \sum_{\substack{n_1 = n_5 - n_6 + n_7 \\ n_6 \neq n_5, n_7}} \widehat{u}_5(n_5) \overline{\widehat{u}_6(n_6)} \widehat{u}_7(n_7) \overline{\widehat{u}_2(n_2)} \widehat{u}_3(n_3) \overline{\widehat{u}_4(n_4)}(t) dt. \end{aligned}$$

Here, we choose  $K$  to be a positive integer such that  $2^K \sim \max(m_1^*, n_1^*)$  unless otherwise stated. In the following, we only consider the terms with  $\mathbf{1}_{[0, T]}(t) \gamma(2^{[\alpha K]}t - m) = \gamma(2^{[\alpha K]}t - m)$  and thus we drop the sharp cut-off  $\mathbf{1}_{[0, T]}(t)$  on the left-hand side. Note that there are  $O(1)$  many values of  $m$  such that  $\mathbf{1}_{[0, T]}(t) \gamma(2^{[\alpha K]}t - m) \neq \gamma(2^{[\alpha K]}t - m)$  and they are easier to handle. See the proof of Proposition 7.3.

With  $f_i = \gamma(2^{[\alpha K]}t - m)u_i$ ,<sup>5</sup>  $2 \leq i \leq 7$ , let  $f_{i,j_i} = \mathcal{F}^{-1}[\eta_{j_i}(\tau - n^2)\widehat{f}_i]$ . Then, it suffices to prove

$$\begin{aligned} &\left| \int_{\mathbb{R}} \sum_{j_2, \dots, j_7} \sum_{|m| \leq T2^{[\alpha K]}} \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ n_2 \neq n_1, n_3 \\ \max |n_j| > M}} \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} \mathbf{1}_{I_{k_1}}(n_1) \sum_{\substack{n_1 = n_5 - n_6 + n_7 \\ n_6 \neq n_5, n_7}} \widehat{f}_{5,j_5}(n_5) \overline{\widehat{f}_{6,j_6}(n_6)} \widehat{f}_{7,j_7}(n_7) \right. \\ &\quad \left. \times \overline{\widehat{f}_{2,j_2}(n_2)} \widehat{f}_{3,j_3}(n_3) \overline{\widehat{f}_{4,j_4}(n_4)}(t) dt \right| \lesssim T^\theta \prod_{i=2}^7 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha} \quad (7.27) \end{aligned}$$

for  $s > -\frac{1}{8}$ . In view of (3.5), we assume that  $j_i \geq \alpha K$ ,  $i = 2, \dots, 7$ . In the following, we prove (7.27) for fixed  $j_i$ ,  $i = 2, \dots, 7$ . Since the summations over  $j_i$  do not play any significant role in the following, we drop the summations over  $j_i$  and we denote  $f_{i,j_i}$  by  $f_i$  for simplicity.

Now, we proceed to prove (7.27). Gathering (7.22), (7.23), (7.24), and (7.25) in the proof of Proposition 7.5, we have the bound

$$\Theta(\bar{n}) := \left| \frac{\Psi_s(\bar{n})}{\Phi(\bar{n})} \right| \lesssim \frac{(n_4^*)^{2s}}{n_3^* n_1^*}, \quad (7.28)$$

which suffices for many cases in the following. Note that, however, each of (7.22), (7.23), and (7.25) is better than (7.28). Let  $\tilde{\Phi}(\bar{n}) = n_1^2 - n_5^2 + n_6^2 - n_7^2 = 2(n_1 - n_5)(n_1 - n_7)$  as in (7.4). We prove (7.27) by performing case-by-case analysis. By symmetry, we assume  $|n_5| \geq |n_7|$  in the following.

• **Case (a):**  $m_1^* \lesssim n_1^*$ .

We first consider the case  $n_1^* \sim n_3^*$ . From (7.28), we have

$$\sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \lesssim T(n_1^*)^{-2-4s+} (n_4^*)^{2s} \leq T(n_1^*)^{6s-} \quad (7.29)$$

<sup>5</sup>Strictly speaking, we need to use  $f_{i,m}$  instead of  $f_i$ . However, all the estimate below are uniform in  $m$ . Thus, for notational simplicity, we drop the subscript  $m$ .



for  $s > -\frac{1}{5}$ . Then, (7.27) follows from the  $L^6$ -Strichartz estimate (5.3) with (7.29):

$$\begin{aligned} |\mathbf{I}(T)| &\lesssim \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \|f_5 \overline{f_6} f_7\|_{L_{x,t}^2} \|\overline{f_2} f_3 \overline{f_4}\|_{L_{x,t}^2} \\ &\lesssim \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \prod_{i=2}^7 \|f_i\|_{L_{x,t}^6} \\ &\lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha} \end{aligned}$$

for  $s > -\frac{1}{5}$ . In the following, we assume  $n_1^* \gg n_3^*$ .

◦ Subcase (a.i):  $|n_1| \gg n_3^*$ .

In this case, we have  $|n_1| \sim n_1^*$ . In particular, we have  $\min(|n_2|, |n_3|, |n_4|) = n_4^*$ . Then, from (7.28), we have

$$\begin{aligned} \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \cdot (n_1^*)^{-\frac{\alpha}{2}} (n_3^*)^{\frac{1}{2}} &\lesssim T(n_1^*)^{-1-2s} (n_3^*)^{-\frac{1}{2}} (n_4^*)^{2s} \\ &\lesssim T(n_1^*)^{4s-} (n_3^*)^s (n_4^*)^s \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}}, \end{aligned} \quad (7.30)$$

where the penultimate inequality holds for  $s > -\frac{1}{6}$ . Noting that  $\min(|n_2|, |n_4|) \lesssim n_3^* \ll |n_1|$ , it follows from the  $L^6$ -Strichartz estimate (5.3), Lemma 7.2 (a), and (7.30) that

$$\begin{aligned} |\mathbf{I}(T)| &\lesssim \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \|f_5 \overline{f_6} f_7\|_{L_{x,t}^2} \|\overline{f_2} f_3 \overline{f_4}\|_{L_{x,t}^2} \\ &\lesssim \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \cdot (n_1^*)^{-\frac{\alpha}{2}} (n_3^*)^{\frac{1}{2}} \left( \prod_{i=5}^7 \|f_i\|_{L_{x,t}^6} \right) \left( \prod_{i=2}^4 \|u_i\|_{F_{k_i}^\alpha} \right) \\ &\lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha} \end{aligned}$$

for  $s > -\frac{1}{6}$ .

◦ Subcase (a.ii):  $|n_1| \lesssim n_3^* \ll n_1^*$ .

Subsubcase (a.ii.1):  $m_1^* \sim m_2^* \gg |n_1|$ . First suppose that  $m_1^* \gtrsim (n_1^*)^\alpha$ .

(i) Suppose  $|n_1| \sim n_4^*$ . Then, from (7.28), we have

$$\sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \cdot (n_1^*)^{-\alpha} n_4^* \lesssim T(n_1^*)^{-1} (n_3^*)^{-1} (n_4^*)^{1+2s} \lesssim T(n_1^*)^{5s-} (n_3^*)^s \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}}$$

for  $s > -\frac{1}{5}$ .

(ii) Suppose  $|n_1| \sim n_3^* \gg n_4^*$ . Then, from (7.28), we have

$$\sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \cdot (n_1^*)^{-\alpha} n_3^* \lesssim T(n_1^*)^{-1} (n_4^*)^{2s} \lesssim T(n_1^*)^{5s-} (n_4^*)^s \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}}$$

for  $s > -\frac{1}{5}$ .

Noting that  $\max(|n_2|, |n_4|) \sim n_1^* \gg n_3^* \gtrsim |n_1|$ , we apply Lemma 7.2 (a) on  $f_5 \overline{f_6} f_7$  and  $\overline{f_2} f_3 \overline{f_4}$ . This yields (7.27) for  $s > -\frac{1}{5}$ .

Next, suppose that  $m_1^* \ll (n_1^*)^\alpha$ . In this case, by applying Lemma 7.2 (a) on  $f_5 \overline{f_6} f_7$ , we only gain the factor of  $(m_1^*)^{-\frac{1}{2}}$  instead of  $(n_1^*)^{-\frac{\alpha}{2}}$ . Nonetheless, (7.27) for  $s > -\frac{1}{6}$  follows from applying Lemma 7.2 (a) on  $f_5 \overline{f_6} f_7$  and  $\overline{f_2} f_3 \overline{f_4}$ , since

$$\begin{aligned} \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \cdot (n_1^*)^{-\frac{\alpha}{2}} (m_1^*)^{-\frac{1}{2}} n_3^* &\lesssim T(n_1^*)^{-1-2s+} (n_4^*)^{2s} (m_1^*)^{-\frac{1}{2}} \\ &\lesssim T(n_1^*)^{3s-} (m_1^*)^{3s} \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \end{aligned}$$

for  $s > -\frac{1}{6}$ .

Subsubcase (a.ii.2):  $m_1^* \sim m_2^* \lesssim |n_1| \lesssim n_3^*$ . From (7.28), we have

$$\sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \lesssim T(n_1^*)^{-1-4s+} (n_3^*)^{-1} (n_4^*)^{2s} \leq T(n_1^*)^{2s-} (n_3^*)^{4s} \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}}$$

for  $s > -\frac{1}{6}$ . Then, (7.27) follows from the  $L^6$ -Strichartz estimate (5.3).

• **Case (b):**  $m_1^* \gg n_1^*$ ,  $|\widetilde{\Phi}(\bar{n})| \lesssim |\Phi(\bar{n})|$ .

In the following, we choose  $K$  such that  $2^K \sim m_1^*$ , unless otherwise stated.

◦ Subcase (b.i):  $|n_1| \lesssim n_3^*$ .

Subsubcase (b.i.1):  $n_1^* \sim n_3^*$ . If  $m_3^* \gg |n_1|$ , then we have

$$m_1^* m_3^* \lesssim |\widetilde{\Phi}(\bar{n})| \lesssim |\Phi(\bar{n})| \lesssim (n_1^*)^2. \quad (7.31)$$

This in particular implies that  $|n_1| = n_4^* \ll n_1^*$ . Otherwise, i.e. if  $|n_1| \sim n_3^* \sim n_1^*$ , then it follows from (7.31) that

$$m_1^* n_1^* \ll m_1^* m_3^* \lesssim (n_1^*)^2,$$

which is a contradiction to the assumption  $m_1^* \gg n_1^*$ .

From (7.31), we have  $m_1^* m_2^* m_3^* \lesssim (n_1^*)^4$  and  $m_1^* \lesssim (n_1^*)^2$ . Then, from (7.28) with  $n_1^* \sim n_3^*$ , we have

$$\begin{aligned} \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \cdot (m_1^*)^{-\frac{\alpha}{2}} (n_4^*)^{\frac{1}{2}} &\lesssim T(n_1^*)^{-\frac{3}{2}+2s} (m_1^*)^{-2s+} \lesssim T(n_1^*)^{3s} (m_1^* m_2^* m_3^*)^s (n_1^*)^{-\frac{3}{2}-9s+} \\ &\lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \end{aligned}$$

for  $s > -\frac{1}{6}$ . Noting that  $\max(|n_5|, |n_7|) \sim m_1^*$ , we apply Lemma 7.2 (a) on  $f_5 \overline{f_6} f_7$ . Then, with the  $L^6$ -Strichartz estimate (5.3), we have

$$\begin{aligned} |I(T)| &\lesssim \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \|f_5 \overline{f_6} f_7\|_{L_{x,t}^2} \|\overline{f_2} f_3 \overline{f_4}\|_{L_{x,t}^2} \\ &\lesssim \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \cdot (m_1^*)^{-\frac{\alpha}{2}} (n_4^*)^{\frac{1}{2}} \left( \prod_{i=5}^7 \|u_i\|_{F_{k_i}^\alpha} \right) \left( \prod_{i=2}^4 \|f_i\|_{L_{x,t}^6} \right) \\ &\lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha} \end{aligned}$$

for  $s > -\frac{1}{6}$ .

Next, suppose that  $m_3^* \lesssim |n_1|$ . Then, we have  $m_1^* \lesssim |\tilde{\Phi}(\bar{n})| \lesssim |\Phi(\bar{n})| \lesssim (n_1^*)^2$ . In particular, we have  $m_1^* m_2^* m_3^* \lesssim (n_1^*)^4 |n_1|$ . In this case, we choose  $K$  such that  $2^K \sim n_1^*$ , although  $n_1^* \ll m_1^*$ . We estimate  $\|f_5 \overline{f_6} f_7\|_{L_{x,t}^2}$  by further dividing the interval of length  $\sim 2^{-[\alpha K]} \sim (n_1^*)^{-\alpha}$  into  $O((m_1^*)^\alpha (n_1^*)^{-\alpha})$ -many subintervals of length  $\sim (m_1^*)^{-\alpha}$ . By the almost orthogonality of the contributions from these subintervals under the  $L_t^2$ -norm, this adds an extra factor of  $(m_1^*)^{\frac{\alpha}{2}} (n_1^*)^{-\frac{\alpha}{2}}$ .

(i) Suppose that  $|n_1| \sim n_4^*$ . Then, from (7.28), we have

$$\begin{aligned} (m_1^*)^{\frac{\alpha}{2}} (n_1^*)^{-\frac{\alpha}{2}} \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \cdot (m_1^*)^{-\frac{\alpha}{2}} (n_4^*)^{\frac{1}{2}} &\lesssim T (n_1^*)^{-2-2s} (n_4^*)^{\frac{1}{2}+2s} \\ &\lesssim T (n_1^*)^{-\frac{3}{2}+} \lesssim T (n_1^*)^{-\frac{3}{2}-8s+} \prod_{i=2}^7 2^{(s-)^{k_i}} \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \end{aligned}$$

for  $s > -\frac{3}{16}$ .

(ii) Suppose that  $|n_1| \gg n_4^*$ . Then, from (7.28), we have

$$\begin{aligned} (m_1^*)^{\frac{\alpha}{2}} (n_1^*)^{-\frac{\alpha}{2}} \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \cdot (m_1^*)^{-\frac{\alpha}{2}} (n_3^*)^{\frac{1}{2}} &\lesssim T (n_1^*)^{-\frac{3}{2}-2s+} (n_4^*)^{2s} \\ &\lesssim T (n_1^*)^{-\frac{3}{2}-9s+} (n_4^*)^s \prod_{i=2}^7 2^{(s-)^{k_i}} \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \end{aligned}$$

for  $s > -\frac{1}{6}$ .

In both cases, we obtain (7.27) by applying Lemma 7.2 (a) on  $f_5 \overline{f_6} f_7$  and the  $L^6$ -Strichartz estimate (5.3) on  $\overline{f_2} f_3 \overline{f_4}$ .

Subsubcase (b.i.2): Next, we consider the case  $n_1^* \gg n_3^*$ . Note that we have  $m_1^* \lesssim (n_1^*)^2$ . Moreover, we have either  $m_3^* \sim |n_1|$  or  $m_1^* m_3^* \lesssim (n_1^*)^2$ . In either case, we have

$$m_1^* m_2^* m_3^* \lesssim (n_1^*)^4 |n_1|.$$

In this case, we choose  $K$  such that  $2^K \sim n_1^*$ , although  $n_1^* \ll m_1^*$ . As in Subsubcase (b.i.1), we lose an extra factor of  $(m_1^*)^{\frac{\alpha}{2}} (n_1^*)^{-\frac{\alpha}{2}}$  by estimating the contribution of  $\|f_5 \overline{f_6} f_7\|_{L_{x,t}^2}$  over subintervals of length  $\sim (m_1^*)^{-\alpha}$ .

(i)  $|n_4 - n_1| \sim n_1^* \gg |n_4 - n_3| = |n_1 - n_2|$ . Since  $|n_1| \lesssim n_3^* \ll n_1^*$ , it follows that  $|n_2| \lesssim n_3^* \ll n_1^*$ . From (7.28), we have

$$(m_1^*)^{\frac{\alpha}{2}}(n_1^*)^{-\frac{\alpha}{2}} \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \cdot (m_1^*)^{-\frac{\alpha}{2}}(n_1^*)^{-\frac{\alpha}{2}}(n_3^*)^{\frac{1}{2}}(n_4^*)^{\frac{1}{2}} \lesssim T(n_1^*)^{-1-2s+}(n_3^*)^{-\frac{1}{2}}(n_4^*)^{\frac{1}{2}+2s}$$

$$\lesssim T(n_1^*)^{-1-6s+}(n_3^*)^{-\frac{1}{2}-s}(n_4^*)^{\frac{1}{2}+s} \prod_{i=2}^7 2^{(s-)^{k_i}} \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}}$$

for  $s > -\frac{1}{6}$ . Then, (7.27) follows from applying Lemma 7.2 (a) on  $\overline{f_5 f_6} f_7$  and  $\overline{f_2 f_3} \overline{f_4}$ .

(ii)  $|n_4 - n_1|, |n_4 - n_3| \sim n_1^*$ . From (7.25), we have

$$(m_1^*)^{\frac{\alpha}{2}}(n_1^*)^{-\frac{\alpha}{2}} \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) \cdot (m_1^*)^{-\frac{\alpha}{2}}(n_1^*)^{-\frac{\alpha}{2}}(n_3^*)^{\frac{1}{2}}(n_4^*)^{\frac{1}{2}} \lesssim T(n_1^*)^{-2-2s+}(n_3^*)^{\frac{1}{2}}(n_4^*)^{\frac{1}{2}+2s}$$

$$\lesssim T(n_1^*)^{-2-6s}(n_3^*)^{\frac{1}{2}-s}(n_4^*)^{\frac{1}{2}+s} \prod_{i=2}^7 2^{(s-)^{k_i}} \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}}$$

for  $s > -\frac{1}{6}$ . Then, (7.27) follows from applying Lemma 7.2 (a) on  $\overline{f_5 f_6} f_7$  and Lemma 7.2 (b) on  $\overline{f_2 f_3} \overline{f_4}$ .

◦ Subcase (b.ii):  $|n_1| \sim n_1^* \gg n_3^*$ .

Note that we have  $|n_1 - n_7| \ll |n_1|$ . Otherwise, i.e. if  $|n_1 - n_7| \gtrsim |n_1|$ , then we have  $|\tilde{\Phi}(\bar{n})| \gtrsim m_1^* n_1^* \gg (n_1^*)^2 \gtrsim |\Phi(\bar{n})|$ , which is a contradiction. Hence, we have  $|n_1 - n_7| \ll |n_1|$ . In particular,  $m_1^* \sim m_2^* \gg m_3^* \sim n_1^*$ .

Subsubcase (b.ii.1):  $|n_4 - n_1| \sim n_1^* \gg |n_4 - n_3| \sim n_3^*$ . From (7.3) and (7.28), we have

$$\Theta(\bar{n}) \lesssim |\tilde{\Phi}(\bar{n})|^{-\gamma} (n_1^* n_3^*)^{-(1-\gamma)} (n_4^*)^{2s}$$

for any  $\gamma \in [0, 1]$ . With  $|\tilde{\Phi}(\bar{n})| \sim m_1^* |n_1 - n_7| \lesssim |\Phi(\bar{n})|$ , we have

$$\begin{aligned} \Theta(\bar{n}) \cdot (m_1^*)^{\frac{\alpha}{4}}(n_4^*)^{\frac{1}{2}} &\lesssim |\tilde{\Phi}(\bar{n})|^{-\frac{3}{4}-s} (m_1^*)^{-s+} (n_1^* n_3^*)^{-\frac{1}{4}+s} (n_4^*)^{\frac{1}{2}+2s} \\ &\leq |n_5 - n_6|^{-\frac{3}{4}-s} (m_1^*)^{-\frac{3}{4}-2s+} (n_1^*)^{2s} (n_3^*)^s (n_4^*)^s \\ &\leq |n_5 - n_6|^{-\frac{3}{4}-s} (m_1^*)^{2s-} (n_1^*)^{2s} (n_3^*)^s (n_4^*)^s \\ &\lesssim |n_5 - n_6|^{-\frac{3}{4}-s} \prod_{i=2}^7 2^{(s-)^{k_i}} \end{aligned} \tag{7.32}$$

for  $s > -\frac{3}{16}$ .

Fix  $n_6 \in \mathbb{Z}$ . Then, by Cauchy-Schwarz inequality, we have

$$\sum_{n_5 \neq n_6} \frac{g(n_5)}{|n_5 - n_6|^\beta} \lesssim \|g\|_{\ell_n^2}$$

for  $\beta > \frac{1}{2}$ . Combining this with the proof, in particular (5.8), of Case (a) of Lemma 5.3 under  $\{n_5, n_6, n_7, n_1\} \leftrightarrow \{n_1, n_2, n_3, n_4\}$ , we obtain

$$\left\| \sum_{\substack{n_5, n_6 \\ n_6 \neq n_5, n_7}} \frac{\widehat{f_5}(n_5, t) \widehat{f_6}(n_6, t) \widehat{f_7}(n_1 - n_5 + n_6, t)}{|n_5 - n_6|^\beta} \right\|_{L_t^2 \ell_{n_1}^2} \lesssim (m_1^*)^{-\frac{\alpha}{2}} \prod_{i=5}^7 \|u_i\|_{F_{k_i}^\alpha}. \tag{7.33}$$

Also, by  $L_{x,t}^4, L_{x,t}^4, L_{x,t}^\infty$ -Hölder inequality, we have

$$\|\overline{f_2 f_3 f_4}\|_{L_{x,t}^2} \lesssim (m_1^*)^{-\frac{\alpha}{4}} (n_4^*)^{\frac{1}{2}} \prod_{i=2}^7 \|u_i\|_{F_{k_i}^\alpha}. \quad (7.34)$$

Hence, the desired estimate (7.27) follows from (7.32), (7.33) (with  $\beta = \frac{3}{4} + s$ ), and (7.34).

Subsubcase (b.ii.2):  $|n_4 - n_1| \sim n_1^* \gg n_3^* \gg |n_4 - n_3|$ . From (7.14) and (7.15), we have

$$\Theta(\bar{n}) \lesssim |\tilde{\Phi}(\bar{n})|^{-\gamma} (n_1^*)^{-(1-\gamma)} (n_3^*)^{2s-1} |n_4 - n_3|^\gamma \ll |\tilde{\Phi}(\bar{n})|^{-\gamma} (n_1^*)^{-(1-\gamma)} (n_3^*)^{2s-1+\gamma}$$

for any  $\gamma \in [0, 1]$ . Then, with  $\gamma = \frac{9}{14}$ , we have

$$\begin{aligned} \Theta(\bar{n}) \cdot (n_3^*)^{\frac{1}{2}} &\lesssim |n_5 - n_6|^{-\frac{9}{14}} (m_1^*)^{-\frac{9}{14}} (n_1^*)^{-\frac{5}{14}} (n_3^*)^{\frac{1}{7}+2s} \\ &\leq |n_5 - n_6|^{-\frac{9}{14}} (m_1^*)^{3s-} (n_1^*)^s (n_3^*)^{2s} \\ &\lesssim |n_5 - n_6|^{-\frac{9}{14}} \prod_{i=2}^7 2^{(s- )k_i} \end{aligned}$$

for  $s > -\frac{3}{14}$ . Then, the rest follows from (7.33) and Lemma 7.2 (a) on  $\overline{f_2 f_3 f_4}$ .

In the following, we consider the remaining case:

$$m_1^* \gg n_1^* \quad \text{and} \quad |\tilde{\Phi}(\bar{n})| \gg |\Phi(\bar{n})|.$$

With  $\sigma_j = \tau_j - n_j^2$ , we have

$$\sigma_1 - \sigma_5 + \sigma_6 - \sigma_7 = -\tilde{\Phi}(\bar{n}), \quad \sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 = \Phi(\bar{n}).$$

Hence, we conclude that  $\max(|\sigma_2|, |\sigma_3|, |\sigma_4|, |\sigma_5|, |\sigma_6|, |\sigma_7|) \gtrsim |\tilde{\Phi}(\bar{n})|$ .

• **Case (c):**  $\max(|\sigma_2|, |\sigma_3|, |\sigma_4|) \gtrsim |\tilde{\Phi}(\bar{n})|$ .

◦ Subcase (c.i):  $|n_7| \sim m_3^* \gg |n_1|$ .

We have  $|\tilde{\Phi}(\bar{n})| \gtrsim m_1^* m_3^*$ . In this case, we set  $2^K \sim n_1^*$ , although  $n_1^* \ll m_1^*$ . By duality (in  $x$ ) and Cauchy-Schwarz inequality, we have

$$\|\mathbf{P}_{k_1}(f_5 \overline{f_6 f_7})\|_{L_t^\infty L_x^2} \lesssim 2^{\frac{k_1}{2}} (m_3^*)^{\frac{1}{2}} \prod_{i=5}^7 \|f_i\|_{L_t^\infty L_x^2}.$$

Then, by Lemma 3.2, we have

$$\begin{aligned} |\mathbf{I}(T)| &\lesssim \sum_{|m| \leq T 2^{\lfloor \alpha K \rfloor}} \Theta(\bar{n}) 2^{\frac{k_1}{2}} (m_3^*)^{\frac{1}{2}} \left( \prod_{i=5}^7 \|f_i\|_{L_t^\infty L_x^2} \right) \|\overline{f_2 f_3 f_4}\|_{L_t^1 L_x^2} \\ &\lesssim T (n_1^*)^{-1+\alpha} (n_3^*)^{-1} (n_4^*)^{2s} 2^{\frac{k_1}{2}} (m_3^*)^{\frac{1}{2}} \left( \prod_{i=5}^7 \|f_i\|_{F_{k_i}^\alpha} \right) \|\overline{f_2 f_3 f_4}\|_{L_t^1 L_x^2}. \end{aligned} \quad (7.35)$$

By Hölder inequality, we place the factor in  $\overline{f_2 f_3 f_4}$  with the highest modulation in  $L_t^2$  and others in  $L_t^2$  and  $L_t^\infty$ , while we place the factors with two low spatial frequencies in  $L_x^\infty$  and

the other in  $L_x^2$ . For example, if  $k_2 \geq k_3 \geq k_4$  and  $j_2 \geq j_3 \geq j_4$ , we have

$$\begin{aligned} \|\overline{f_2 f_3 f_4}\|_{L_t^1 L_x^2} &\leq \|f_2\|_{L_{x,t}^2} \|f_3\|_{L_t^2 L_x^\infty} \|f_4\|_{L_{x,t}^\infty} \\ &\lesssim (m_1^* m_3^*)^{-\frac{1}{2}} (n_1^*)^{-\frac{\alpha}{2}} 2^{\frac{k_3}{2}} 2^{\frac{k_4}{2}} \prod_{i=2}^4 \|u_i\|_{F_{k_i}^\alpha}. \end{aligned} \quad (7.36)$$

Here, we used the fact that  $2^{j_2} \gtrsim m_1^* m_3^*$  and  $2^{j_3} \gtrsim (n_1^*)^\alpha$ . By collecting the weights from (7.35) and (7.36), we have

$$\begin{aligned} (m_1^*)^{-\frac{1}{2}} (n_1^*)^{-1-2s} (n_3^*)^{-1} (n_4^*)^{2s} 2^{\frac{k_1+k_3+k_4}{2}} \\ \lesssim (m_1^*)^{-\frac{1}{2}} (n_1^*)^{-\frac{1}{2}-2s} (n_3^*)^{2s} \lesssim (m_1^*)^{3s-} (n_1^*)^{2s} (n_3^*)^s, \end{aligned} \quad (7.37)$$

where the last inequality holds as long as  $s > -\frac{1}{8}$ . Hence, (7.27) holds for  $s > -\frac{1}{8}$  in this case. Note that the same proof holds when  $|n_5| \sim |n_7| \sim m_1^*$ , since we have  $|\tilde{\Phi}(\bar{n})| \sim (m_1^*)^2$  in this case.

◦ Subcase (c.ii):  $|n_7| \sim m_3^* \ll |n_1|$ .

In this case, we have  $|\tilde{\Phi}(\bar{n})| \gtrsim m_1^* |n_1| \gg m_1^* m_3^*$ . Proceeding as in Subcase (c.i), we obtain (7.27) for  $s > -\frac{1}{8}$ .

◦ Subcase (c.iii):  $|n_7| \sim |n_1|$ .

In this case, we first estimate the contribution from  $|n_7 - n_1| \sim 2^L$  for each  $L$  and sum over  $L$  in the end. Note that  $L \leq k_1 + 10$ . When  $|n_7 - n_1| \sim 2^L$ , we have

$$|\tilde{\Phi}(\bar{n})| \sim 2^L m_1^*. \quad (7.38)$$

Moreover, we have  $|n_5 - n_6| \sim 2^L$ .

As in the proof of Lemma 5.5, write  $I_{k_i} = \bigcup_{\ell_i} J_{i\ell_i}$ ,  $i = 5, 6$ , where  $|J_{i\ell_i}| \sim 2^L$ . Then, if  $n_5 \in J_{5\ell_5}$  for some  $\ell_5$ , there are  $O(1)$  many possible values for  $\ell_6 = \ell_6(\ell_5)$  such that  $n_6 \in J_{6\ell_6}$ . Then, by writing

$$\sum_{n_5} \sum_{n_6} = \sum_{\ell_5} \sum_{\ell_6=\ell_6(\ell_5)} \sum_{n_5 \in J_{5\ell_5}} \sum_{n_6 \in J_{6\ell_6}},$$

we only lose  $2^L$  by applying Cauchy-Schwarz inequality in  $n_5$  and  $n_6$ . Then, by Young's inequality, we have

$$\begin{aligned} \|f_5 \overline{f_6} f_7\|_{L_t^\infty L_x^2} &\lesssim \sum_{\ell_5} \sum_{\ell_6=\ell_6(\ell_5)} \|\widehat{f}_{5,\ell_5}\|_{L_t^\infty \ell_n^1} \|\widehat{f}_{6,\ell_6(\ell_5)}\|_{L_t^\infty \ell_n^1} \|\widehat{f}_7\|_{L_t^\infty \ell_n^2} \\ &\lesssim 2^L \prod_{i=5}^7 \|f_i\|_{L_t^\infty L_x^2}. \end{aligned} \quad (7.39)$$

In the last step, after Cauchy-Schwarz inequality in  $n_5$  and  $n_6$ , we also applied Cauchy-Schwarz inequality in  $\ell_5$  to sum the contribution from  $\widehat{f}_{5,\ell_5}(n, t) := \mathbf{1}_{J_{5\ell_5}}(n) \cdot \widehat{f}_5(n, t)$  and  $\widehat{f}_{6,\ell_6(\ell_5)}$  over  $\ell_5$ .

As before, we set  $2^K \sim n_1^*$ . Without loss of generality, assume  $k_2 \geq k_3 \geq k_4$ . Then, by proceeding as in Subcase (c.i) with (7.38) and (7.39), we have

$$\begin{aligned} |I(T)| &\lesssim \sum_{L=0}^{k_1+10} T \Theta(\bar{n}) |\tilde{\Phi}(\bar{n})|^{-\frac{1}{2}} (n_1^*)^{-2s+2L} 2^{\frac{k_3+k_4}{2}} \prod_{i=2}^7 \|u_i\|_{F_{k_i}^\alpha} \\ &\lesssim T (m_1^*)^{-\frac{1}{2}} (n_1^*)^{-1-2s} (n_3^*)^{-1} (n_4^*)^{2s} 2^{\frac{k_1+k_3+k_4}{2}} \prod_{i=2}^7 \|u_i\|_{F_{k_i}^\alpha}. \end{aligned}$$

Hence, by (7.37), we obtain (7.27) for  $s > -\frac{1}{8}$ .

• **Case (d):**  $\max(|\sigma_5|, |\sigma_6|, |\sigma_7|) \gtrsim |\tilde{\Phi}(\bar{n})|$ .

In this case, we set  $K$  by  $2^K \sim m_1^*$ . Recall that we assume  $|n_5| \geq |n_7|$ . In particular, we have  $|n_5| \sim \max(|n_6|, |n_7|) \sim m_1^*$ .

◦ Subcase (d.i):  $|n_7| \sim m_3^* \ll |n_1|$ .

In this case, we have  $|\tilde{\Phi}(\bar{n})| \gtrsim 2^{k_1} m_1^*$ . Write  $I_{k_i}^* = \bigcup_{\ell_i} J_{i\ell_i}$  with  $|J_{i\ell_i}| \sim n_3^*$ ,  $i = 1, 2$ . Then, with  $n_1 - n_2 + n_3 - n_4 = 0$ , write the terms corresponding to  $n_1^*$  and  $n_2^*$  as sums of their restrictions on  $J_{i\ell_i}$  as in Subcase (c.iii).

(i)  $|n_1| \sim n_4^*$ . Proceeding as in (7.39), we have

$$\|\mathbf{P}_{k_1}(\overline{f_2 f_3 f_4})\|_{L_{x,t}^\infty} \lesssim (n_4^*)^{\frac{1}{2}} \|\overline{f_2 f_3 f_4}\|_{L_t^\infty L_x^2} \lesssim n_3^* (n_4^*)^{\frac{1}{2}} \prod_{i=2}^4 \|f_i\|_{L_t^\infty L_x^2}. \quad (7.40)$$

(ii)  $|n_1| \gg n_4^*$ . By Young's inequality followed by Cauchy-Schwarz inequality, we have

$$\|\overline{f_2 f_3 f_4}\|_{L_{x,t}^\infty} \lesssim \|\mathcal{F}_x(\overline{f_2 f_3 f_4})\|_{L_t^\infty \ell_n^1} \lesssim n_3^* (n_4^*)^{\frac{1}{2}} \prod_{i=2}^4 \|f_i\|_{L_t^\infty L_x^2}. \quad (7.41)$$

Now, we can basically proceed as in Subcase (c.i), after switching the indices  $\{5, 6, 7\} \leftrightarrow \{2, 3, 4\}$ . With (7.40), (7.41), (7.28), and Lemma 3.2, we have

$$\begin{aligned} |I(T)| &\lesssim \sum_{|m| \leq T 2^{\lfloor \alpha K \rfloor}} \Theta(\bar{n}) n_3^* (n_4^*)^{\frac{1}{2}} \left( \prod_{i=2}^4 \|f_i\|_{L_t^\infty L_x^2} \right) \|f_5 \overline{f_6 f_7}\|_{L_{x,t}^1} \\ &\lesssim \sum_{|m| \leq T 2^{\lfloor \alpha K \rfloor}} \Theta(\bar{n}) n_3^* (n_4^*)^{\frac{1}{2}} |\tilde{\Phi}(\bar{n})|^{-\frac{1}{2}} (m_1^*)^{-\frac{\alpha}{2}} (m_3^*)^{\frac{1}{2}} \prod_{i=2}^7 \|f_i\|_{F_{k_i}^\alpha} \\ &\lesssim T (n_1^*)^{-\frac{1}{2}+2s} (m_1^*)^{-\frac{1}{2}-2s} \prod_{i=2}^7 \|u_i\|_{F_{k_i}^\alpha} \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha}, \end{aligned}$$

where the last inequality holds for  $s > -\frac{1}{8}$ .

◦ Subcase (d.ii):  $|n_7| \sim |n_1|$ .

In this case, we first estimate the contribution from  $|n_7 - n_1| \sim 2^L$  for each  $L \leq k_1 + 10$  as in Subcase (c.iii). When  $|n_1| \sim n_4^*$ , it follows from Young's inequality that

$$\|\mathbf{P}_{k_1}(\overline{f_2 f_3 f_4})\|_{L_t^\infty L_x^2} \lesssim (n_4^*)^{\frac{1}{2}} \|\mathcal{F}_x(\overline{f_2 f_3 f_4})\|_{L_t^\infty \ell_n^\infty} \lesssim (n_3^*)^{\frac{1}{2}} (n_4^*)^{\frac{1}{2}} \prod_{i=2}^4 \|f_i\|_{L_t^\infty L_x^2}. \quad (7.42)$$

When  $|n_1| \sim n_3^*$ , a similar computation yields (7.42), while (7.42) trivially follows when  $|n_1| \sim n_1^*$ .

With (7.39) (where  $L_t^\infty$  on the left-hand side is replaced by  $L_t^1$ ) and (7.42), we have

$$\begin{aligned} |\mathbf{I}(T)| &\lesssim \sum_{L=0}^{k_1+10} T \Theta(\bar{n}) |\tilde{\Phi}(\bar{n})|^{-\frac{1}{2}} (m_1^*)^{-2s+2L} (n_3^*)^{\frac{1}{2}} (n_4^*)^{\frac{1}{2}} \prod_{i=2}^7 \|u_i\|_{F_{k_i}^\alpha} \\ &\lesssim T (m_1^*)^{-\frac{1}{2}-4s} (n_1^*)^{-\frac{1}{2}-3s} (n_3^*)^{-\frac{1}{2}-s} (n_4^*)^{\frac{1}{2}+2s} \prod_{i=2}^7 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha}. \end{aligned}$$

Hence, (7.27) holds for  $s > -\frac{1}{8}$ .

◦ Subcase (d.iii):  $|n_7| \gg |n_1|$ .

In this case, we have  $|\tilde{\Phi}(\bar{n})| \gtrsim m_1^* m_3^*$ . We first assume that  $m_3^* \lesssim (n_1^*)^3$ . Then, a slight modification of Subcase (d.i) yields

$$\begin{aligned} |\mathbf{I}(T)| &\lesssim \sum_{|m| \leq T 2^{[\alpha K]}} \Theta(\bar{n}) n_3^* (n_4^*)^{\frac{1}{2}} \left( \prod_{i=2}^4 \|f_i\|_{L_t^\infty L_x^2} \right) \|f_5 \bar{f}_6 f_7\|_{L_{x,t}^1} \\ &\lesssim \sum_{|m| \leq T 2^{[\alpha K]}} \Theta(\bar{n}) n_3^* (n_4^*)^{\frac{1}{2}} |\tilde{\Phi}(\bar{n})|^{-\frac{1}{2}} (m_1^*)^{-\frac{\alpha}{2}} (m_3^*)^{\frac{1}{2}} \prod_{i=2}^7 \|f_i\|_{F_{k_i}^\alpha} \\ &\lesssim T (n_1^*)^{-\frac{1}{2}+2s} (m_1^*)^{-\frac{1}{2}-2s} \prod_{i=2}^7 \|u_i\|_{F_{k_i}^\alpha} \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha}, \end{aligned}$$

where the last inequality holds for  $s > -\frac{1}{8}$ .

Next, we consider the case  $m_3^* \gg (n_1^*)^3$ . Suppose  $|\sigma_5| \gtrsim |\tilde{\Phi}(\bar{n})|$ . By Lemma 3.2, we have

$$\|\bar{f}_2 \bar{f}_5 \bar{f}_4\|_{L_{x,t}^2} \leq \|f_5\|_{L_{x,t}^2} \|f_2\|_{L_{x,t}^\infty} \|f_4\|_{L_{x,t}^\infty} \lesssim |\tilde{\Phi}(\bar{n})|^{-\frac{1}{2}} 2^{\frac{k_2+k_4}{2}} \prod_{i=2,4,5} \|f_i\|_{F_{k_i}^\alpha}. \quad (7.43)$$

By Lemma 7.2 (a), we have

$$\|f_3 \bar{f}_6 f_7\|_{L_{x,t}^2} \lesssim 2^{\frac{k_3}{2}} (m_1^*)^{-\frac{\alpha}{2}} \prod_{i=3,6,7} \|f_i\|_{F_{k_i}^\alpha}. \quad (7.44)$$

Indeed, (7.44) follows from Lemma 7.2 (a) under  $\{n_1, n_2, n_3, n_4\} \leftrightarrow \{n_3, n_6, n_7, n\}$ , where  $n$  is the frequency of the duality variable given by  $n = n_3 - n_6 + n_7$ . Here, we used the fact that  $|n| \sim m_1^* \gg |n_3|$ , since  $|n - n_3| = |n_7 - n_6| = |n_5 - n_1| \sim m_1^*$ . Then, with (7.28), we have

$$\begin{aligned} \sum_{|m| \leq T 2^{[\alpha K]}} \Theta(\bar{n}) |\tilde{\Phi}(\bar{n})|^{-\frac{1}{2}} 2^{\frac{k_2+k_3+k_4}{2}} (m_1^*)^{-\frac{\alpha}{2}} \\ \lesssim T (m_1^*)^{-\frac{1}{2}-2s} (m_3^*)^{-\frac{1}{2}} (n_3^*)^{-\frac{1}{2}} (n_4^*)^{2s} \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \end{aligned} \quad (7.45)$$

for  $s > -\frac{1}{8}$ . From (7.43), (7.44), and (7.45), we obtain (7.27) for  $s > -\frac{1}{8}$ .

If  $|\sigma_7| \gtrsim |\tilde{\Phi}(\bar{n})|$  and  $|n_7| \sim m_1^*$ , then the argument follows by switching the indices  $5 \leftrightarrow 7$ . Namely, we apply Lemma 7.2 (a) on  $f_3 \bar{f}_6 f_5$  under  $\{n_1, n_2, n_3, n_4\} \leftrightarrow \{n_3, n_6, n_5, n\}$ , where  $n$  is the frequency of the duality variable given by  $n = n_3 - n_6 + n_5$ . If  $|\sigma_7| \gtrsim |\tilde{\Phi}(\bar{n})|$  and  $|n_7| \sim m_3^* \ll m_1^*$ , then Lemma 7.2 (a) on  $f_3 \bar{f}_6 f_5$  only yields a gain of  $\max((m_1^*)^{-\frac{\alpha}{2}}, (m_3^*)^{-\frac{1}{2}})$ , since  $|n - n_3| = |n_5 - n_6| = |n_7 - n_1| \sim m_3^*$ . In this case, before proceeding as in (7.43) with  $5 \leftrightarrow 7$ ,



we group  $\overline{f_2 f_7 f_4} = \overline{f_{2,m} f_{7,m} f_{4,m}}$ , which are localized on time intervals of length  $\sim (m_1^*)^{-\alpha}$ , into components localized on time intervals of length  $\sim (m_3^*)^{-\alpha}$ . Then, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} |I(T)| &\lesssim \Theta(\bar{n}) \left( \sum_{|m| \leq T2^{[\alpha K]}} \|\overline{f_{2,m} f_{7,m} f_{4,m}}\|_{L_{x,t}^2}^2 \right)^{\frac{1}{2}} \left( \sum_{|m| \leq T2^{[\alpha K]}} \|f_{3,m} \overline{f_{6,m} f_{5,m}}\|_{L_{x,t}^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim T \Theta(\bar{n}) (m_1^* m_3^*)^{\frac{\alpha}{2}} \left( \sup_{\tilde{m}} \|\overline{f_{2,\tilde{m}} f_{7,\tilde{m}} f_{4,\tilde{m}}}\|_{L_{x,t}^2} \right) \left( \sup_m \|f_{3,m} \overline{f_{6,m} f_{5,m}}\|_{L_{x,t}^2} \right). \end{aligned} \quad (7.46)$$

With (7.28), we have

$$\begin{aligned} T \Theta(\bar{n}) (m_1^* m_3^*)^{\frac{\alpha}{2}} |\tilde{\Phi}(\bar{n})|^{-\frac{1}{2}} 2^{\frac{k_2+k_3+k_4}{2}} (m_3^*)^{-\frac{1}{2}} \\ \lesssim T (m_1^*)^{-\frac{1}{2}-2s+} (m_3^*)^{-1-2s+} (n_3^*)^{-\frac{1}{2}} (n_4^*)^{2s} \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \end{aligned} \quad (7.47)$$

for  $s > -\frac{1}{8}$ . Hence, from (7.43) (with  $5 \leftrightarrow 7$ ), Lemma 7.2 (a) on  $\overline{f_3 f_6 f_5}$ , (7.46), and (7.47), we obtain (7.27) for  $s > -\frac{1}{8}$ .

Lastly, suppose that  $|\sigma_6| \gtrsim |\tilde{\Phi}(\bar{n})|$ . When  $|n_5 - n_7| \sim m_1^*$ , we group  $f_j$ 's into  $\overline{f_2 f_3 f_6}$  and  $f_5 \overline{f_4 f_7}$ . Then, we use (a slight modification of) (7.43) and (7.44) for  $\overline{f_2 f_3 f_6}$  and  $f_5 \overline{f_4 f_7}$ , respectively. In particular, the analogue of (7.44) for  $f_5 \overline{f_4 f_7}$  follows from Lemma 7.2 (b) under  $\{n_1, n_2, n_3, n_4\} \leftrightarrow \{n_7, n_4, n_5, n\}$ , where  $n = n_5 - n_4 + n_7$ . Hence, (7.27) holds for  $s > -\frac{1}{8}$ .

When  $|n_5 - n_7| \ll m_1^*$ , it follows from  $|n_5| \sim m_1^*$  and  $|n_5 - n_6 + n_7| = |n_1| \ll m_1^*$  that  $|n_5| \sim |n_6| \sim |n_7| \sim m_1^*$ . In particular, we have  $|\tilde{\Phi}(\bar{n})| \gtrsim (m_1^*)^2$ . Then, a slight modification of Subcase (d.i) with  $L_{x,t}^4, L_{x,t}^2, L_{x,t}^4$ -Hölder's inequality on  $f_5 \overline{f_6 f_7}$  yields

$$\begin{aligned} |I(T)| &\lesssim \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) n_3^* (n_4^*)^{\frac{1}{2}} \left( \prod_{i=2}^4 \|f_i\|_{L_i^\infty L_x^2} \right) \|f_5 \overline{f_6 f_7}\|_{L_{x,t}^1} \\ &\lesssim \sum_{|m| \leq T2^{[\alpha K]}} \Theta(\bar{n}) n_3^* (n_4^*)^{\frac{1}{2}} |\tilde{\Phi}(\bar{n})|^{-\frac{1}{2}} (m_1^*)^{-\frac{\alpha}{4}} \prod_{i=2}^7 \|f_i\|_{F_{k_i}^\alpha} \\ &\lesssim T (m_1^*)^{-1-3s+} (n_1^*)^{-\frac{1}{2}+2s} \prod_{i=2}^7 \|u_i\|_{F_{k_i}^\alpha} \lesssim T \prod_{i=2}^7 2^{(s-)^{k_i}} \|u_i\|_{F_{k_i}^\alpha}, \end{aligned}$$

where the last inequality holds for  $s > -\frac{1}{6}$ .

This completes the proof of Proposition 7.6.  $\square$

**Remark 7.7.** We point out that these energy estimates have a certain smoothing property. From the proofs of Propositions 7.3, 7.5, and 7.6, it is easy to see that the energy estimates (7.8), (7.20), and (7.26) hold even if we replace the  $F^{s,\alpha}(T)$ -norms on the right-hand sides by  $F^{s-\delta,\alpha}(T)$ -norms for some small  $\delta > 0$ . Namely, we can take the regularity on the right-hand sides to be slightly less than that on the left-hand sides.

## 8. EXISTENCE OF SOLUTIONS TO THE WICK ORDERED CUBIC NLS

In this section, we present the proof of Theorem 1.2. First, we establish an a priori bound of smooth solutions to the Wick ordered cubic NLS (1.3). Fix  $s \in (-\frac{1}{8}, 0)$  and

$\alpha = -4s+$ . Let  $u \in C(\mathbb{R}; H^\infty(\mathbb{T}))$  be a smooth global solution to (1.3) with initial condition  $u_0 \in H^\infty(\mathbb{T})$ . Then, by Propositions 4.1, 6.1, 7.3, 7.5, and 7.6, we have<sup>6</sup>

$$\|u\|_{F^{s,\alpha}(T)} \lesssim \|u\|_{E^s(T)} + \|\mathfrak{N}(u)\|_{N^{s,\alpha}(T)}, \quad (8.1)$$

$$\|\mathfrak{N}(u)\|_{N^{s,\alpha}(T)} \lesssim T^\theta \|u\|_{F^{s,\alpha}(T)}^3, \quad (8.2)$$

$$\|u\|_{E^s(T)}^2 \leq \|u_0\|_{H^s}^2 + C \left( T^\theta M^{c(s)} \|u\|_{F^{s,\alpha}(T)}^4 + M^{-d(s)} \|u\|_{F^{s,\alpha}(T)}^4 + T^\theta \|u\|_{F^{s,\alpha}(T)}^6 \right), \quad (8.3)$$

for  $T \in (0, 1]$ ,  $M \in \mathbb{N}$ ,  $c(s) \geq 0$ ,  $d(s) > 0$ , and  $\theta > 0$ , where  $\mathfrak{N}(u) = \mathcal{N}(u) + \mathcal{R}(u)$ . In the following,  $T$  and  $M$  will be chosen in terms of  $\|u_0\|_{H^s}$ .

First, we establish an a priori bound on the smooth solution  $u$  in terms of the  $H^s$ -norm of the initial condition  $u_0$ . Let  $X(T) = \|u\|_{E^s(T)} + \|\mathfrak{N}(u)\|_{N^{s,\alpha}(T)}$ . Then, we have the following lemma.

**Lemma 8.1.** *Let  $u \in C([-1, 1]; H^\infty(\mathbb{T}))$ . Then,  $X(T)$  is non-decreasing and continuous in  $T \in [0, 1]$ . Moreover, we have*

$$\lim_{T \rightarrow 0} X(T) = \|u(0)\|_{H^s}.$$

*Proof.* From the definition,  $X(T)$  is non-decreasing in  $T$ . Let  $I_T = [-T, T]$ . Then, given  $0 \leq T_1 < T_2 \leq 1$ , we have

$$\|u\|_{E^s(T_2)}^2 - \|u\|_{E^s(T_1)}^2 = \sum_{k \geq 1} 2^{2sk} \left( \sup_{t_k \in I_{T_2}} \|\mathbf{P}_k u(t_k)\|_{L^2}^2 - \sup_{\tilde{t}_k \in I_{T_1}} \|\mathbf{P}_k u(\tilde{t}_k)\|_{L^2}^2 \right)$$

By letting  $\mathcal{A} = \{k \geq 1 : \sup_{t_k \in I_{T_2}} \|\mathbf{P}_k u(t_k)\|_{L^2}^2 = \sup_{t_k \in I_{T_2} \setminus I_{T_1}} \|\mathbf{P}_k u(t_k)\|_{L^2}^2\}$ , we have

$$\begin{aligned} &\leq \sum_{k \in \mathcal{A}} 2^{2sk} \left( \sup_{t_k \in I_{T_2} \setminus I_{T_1}} \|\mathbf{P}_k u(t_k)\|_{L^2}^2 - \|\mathbf{P}_k u(T_1)\|_{L^2}^2 \right) \\ &\leq \sup_{t_* \in I_{T_2} \setminus I_{T_1}} \sum_{k \in \mathcal{A}} 2^{2(s+\varepsilon)k} \left( \|\mathbf{P}_k u(t_*)\|_{L^2}^2 - \|\mathbf{P}_k u(T_1)\|_{L^2}^2 \right) \\ &\leq 2 \|u\|_{C([-1, 1]; H^{s+\varepsilon})} \sup_{t_* \in I_{T_2} \setminus I_{T_1}} \|u(t_*) - u(T_1)\|_{H^{s+\varepsilon}} \longrightarrow 0, \end{aligned}$$

as  $T_2 - T_1 \rightarrow 0$ . This shows the continuity of  $\|u\|_{E^s(T)}$ . In particular, we have

$$\lim_{T \rightarrow 0} \|u\|_{E^s(T)} = \|u(0)\|_{H^s}.$$

Let  $Y(T) = \|u\|_{N^{s,\alpha}(T)}$ . Then, with Lemma 3.1, it is easy to see that

$$Y(T) \lesssim T^{\frac{1}{2}} \|u\|_{C([-T, T]; H^s)} \longrightarrow 0,$$

as  $T \rightarrow 0$ . Hence, it remains to prove the continuity of  $Y(T)$ . Let  $T_0 \in (0, 1]$ . First, by the monotonicity in  $T$  of the  $N^{s,\alpha}(T)$ -norm and  $\|u\|_{N^{s,\alpha}((1+\delta)T_0)} < \infty$  (with some small  $\delta > 0$ <sup>7</sup> such that  $(1+\delta)T_0 \leq 1$ ), it follows from Monotone Convergence Theorem that given  $\varepsilon > 0$ , there exists  $K \in \mathbb{N}$  such that  $\|\mathbf{P}_{>K} u\|_{N^{s,\alpha}(T)} < \varepsilon$  for all  $T < (1+\delta)T_0$ . Then, it suffices to show that there exists  $\delta_0 = \delta_0(T_0, \varepsilon) > 0$  such that

$$\|\mathbf{P}_{\leq K} u\|_{N^{s,\alpha}(T_0)} - \|\mathbf{P}_{\leq K} u\|_{N^{s,\alpha}(rT_0)} < \varepsilon \quad (8.4)$$

<sup>6</sup>When  $s > -\frac{1}{10}$ , we do not need the last two terms in (8.3). Moreover, we can choose  $c(s) = 0$  in this case. See Remark 7.4.

<sup>7</sup>Set  $\delta = 0$  when  $T_0 = 1$ .

for all  $r \in \mathbb{R}$  with  $|r - 1| < \delta_0$ . In the following, we restrict our attention to  $\mathbf{P}_{\leq K} u$  for this fixed  $K = K(T_0, \varepsilon)$ . For simplicity, we denote  $\mathbf{P}_{\leq K} u$  by  $u$ , i.e. we assume that the spatial frequencies are supported on  $\{|n| < 2^K\}$  in the following.

As in [24], we introduce the dilation operator  $D_r$  given by  $D_r(u)(x, t) = u(x, r^{-1}t)$  for  $r$  close to 1. Then, with  $r = T/T_0$ , by the triangle inequality and Lemma 3.1, we have

$$\begin{aligned} \left| \|u\|_{N^{s,\alpha}(T)} - \|D_r(u)\|_{N^{s,\alpha}(T)} \right| &\leq \|u - D_r(u)\|_{N^{s,\alpha}(T)} \\ &\lesssim (T)^{\frac{1}{2}} \|u - D_r(u)\|_{L_t^\infty([-T, T]; H^s)} \rightarrow 0 \end{aligned}$$

as  $T \rightarrow T_0$ , i.e. as  $r \rightarrow 1$ , since  $u \in C([-1, 1]; H^\infty(\mathbb{T}))$ . Then, it remains to show that

$$\lim_{r \rightarrow 1} \|D_r(u)\|_{N^{s,\alpha}(rT_0)} = \|u\|_{N^{s,\alpha}(T_0)}. \quad (8.5)$$

Note that (8.5) follows once we show

$$\|u\|_{N^{s,\alpha}(T_0)} \leq \liminf_{r \rightarrow 1} \|D_r(u)\|_{N^{s,\alpha}(rT_0)} \quad (8.6)$$

and

$$\limsup_{r \rightarrow 1} \|D_r(u)\|_{N^{s,\alpha}(rT_0)} \leq \|u\|_{N^{s,\alpha}(T_0)}. \quad (8.7)$$

We first prove (8.6) in the following. Given small  $\tilde{\varepsilon} > 0$ , let  $u_r$  be an extension of  $D_r(u)$  outside  $[-rT_0, rT_0]$ <sup>8</sup> such that

$$\|u_r\|_{N^{s,\alpha}} \leq \|D_r(u)\|_{N^{s,\alpha}(rT_0)} + \tilde{\varepsilon}.$$

Then,  $D_{\frac{1}{r}}(u_r)$  is an extension of  $u$  outside  $[-T_0, T_0]$  and by definition, we have

$$\|u\|_{N^{s,\alpha}(T_0)} \leq \|D_{\frac{1}{r}}(u_r)\|_{N^{s,\alpha}}.$$

Hence, it suffices to prove

$$\|D_{\frac{1}{r}}(u_r)\|_{N^{s,\alpha}} \leq \psi(r) \|u_r\|_{N^{s,\alpha}} \quad (8.8)$$

for some continuous function  $\psi(r)$  defined in a neighborhood of  $r = 1$  such that  $\lim_{r \rightarrow 1} \psi(r) = 1$ .

Fix  $k \in \mathbb{Z}_+ \cap [0, K]$ . With  $\eta_0^r(t) = \eta_0(r^{-1}t)$ , we have

$$\eta_0(2^{[\alpha k]}(t - t_k)) \cdot \mathbf{P}_k D_{\frac{1}{r}}(u_r) = \mathbf{P}_k D_{\frac{1}{r}}(\eta_0^r(2^{[\alpha k]}(t - rt_k)) \cdot u_r).$$

Hence, we have

$$\mathcal{F}[\eta_0(2^{[\alpha k]}(t - t_k)) \cdot \mathbf{P}_k D_{\frac{1}{r}}(u_r)](n, \tau) = r^{-1} \mathbf{1}_{I_k}(n) \mathcal{F}[\eta_0^r(2^{[\alpha k]}(t - rt_k)) \cdot u_r](n, r^{-1}\tau).$$

Then, we have

$$\begin{aligned} \|\mathbf{P}_k D_{\frac{1}{r}}(u_r)\|_{N_k^\alpha} &= \sup_{t_k \in \mathbb{R}} \left\| (\tau - n^2 + i2^{[\alpha k]})^{-1} \mathcal{F}[\eta_0(2^{[\alpha k]}(t - t_k)) \cdot \mathbf{P}_k D_{\frac{1}{r}}(u_r)] \right\|_{X_k} \\ &= r^{-\frac{1}{2}} \sup_{t_k \in \mathbb{R}} \sum_{j=0}^{\infty} 2^{\frac{j}{2}} \left\| \frac{\eta_j(r\tau - n^2)}{|r\tau - n^2 + i2^{[\alpha k]}|} \mathbf{1}_{I_k}(n) \mathcal{F}[\eta_0^r(2^{[\alpha k]}(t - t_k)) \cdot u_r] \right\|_{\ell_n^2 L_\tau^2}. \end{aligned} \quad (8.9)$$

<sup>8</sup>Strictly speaking, we construct extensions  $u_r^k$  for  $\mathbf{P}_k D_r(u)$ ,  $k \in \mathbb{Z}_+ \cap [0, K]$  and then construct  $u_r$  by setting  $\mathbf{P}_k(u_r) := u_r^k$ . By construction, we have  $\mathbf{P}_{\leq K}(u_r) = u_r$ .

We first examine the integrand in (8.9). If  $|\tau| \gg n^2$ , we have  $|\tau| \sim |\tau - n^2| \sim |r\tau - n^2| \sim 2^j$  for  $r$  sufficiently close to 1 on the support of  $\eta_j(r\tau - n^2)$ . Otherwise, we have  $|\tau| \lesssim n^2 \sim 2^{2k}$ . Thus, we have

$$\left| \frac{1}{(r\tau - n^2)^2 + 2^{2[\alpha k]}} - \frac{1}{(\tau - n^2)^2 + 2^{2[\alpha k]}} \right| = \left| \frac{(1-r)\tau((1+r)\tau - 2n^2)}{((r\tau - n^2)^2 + 2^{2[\alpha k]})(\tau - n^2)^2 + 2^{2[\alpha k]}} \right| \lesssim |r-1| 2^{4k} \frac{1}{(\tau - n^2)^2 + 2^{2[\alpha k]}}. \quad (8.10)$$

On the other hand, by Mean Value Theorem, we have

$$\begin{aligned} |\eta_j(r\tau - n^2) - \eta_j(\tau - n^2)| &\lesssim |r-1| \sup_{s \in J_r} 2^{-j} |\tau| |\eta'(2^{-j}(s\tau - n^2))| \\ &\lesssim \begin{cases} |r-1| \sum_{|j'-j| \leq 1} \eta_{j'}(\tau - n^2), & \text{if } j \geq 2k+5, \\ |r-1| 2^{-j} 2^{2k} \sum_{j' \leq 2k+10} \eta_{j'}(\tau - n^2), & \text{otherwise,} \end{cases} \end{aligned} \quad (8.11)$$

for  $r$  sufficiently close to 1, where  $J_r = [1, r]$  if  $r > 1$  and  $[r, 1]$  if  $r < 1$ .

From (8.9), (8.10), and (8.11), we have

$$\begin{aligned} \|\mathbf{P}_k D_{\frac{1}{r}}(u_r)\|_{N_k^\alpha} &\leq \varphi_0(r) \sup_{t_k \in \mathbb{R}} \sum_{j=0}^{\infty} 2^{\frac{j}{2}} \left\| \frac{\eta_j(\tau - n^2)}{|\tau - n^2 + i2^{[\alpha k]}|} \mathbf{1}_{I_k}(n) \mathcal{F}[\eta_0^r(2^{[\alpha k]}(t - t_k)) \cdot u_r] \right\|_{\ell_n^2 L_\tau^2} \\ &\quad + \mathbf{I}_k + \mathbf{II}_k + \mathbf{III}_k, \end{aligned} \quad (8.12)$$

where  $\varphi_0(r) = r^{-\frac{1}{2}}(1 + c|r-1|) \rightarrow 1$  as  $r \rightarrow 1$ , and  $\mathbf{I}_k, \mathbf{II}_k, \mathbf{III}_k$  are given by

$$\begin{aligned} \mathbf{I}_k &\sim \varphi_1(r) 2^{2k} \sup_{t_k \in \mathbb{R}} \sum_{j=0}^{\infty} 2^{\frac{j}{2}} \left\| \frac{\eta_j(\tau - n^2)}{|\tau - n^2 + i2^{[\alpha k]}|} \mathbf{1}_{I_k}(n) \mathcal{F}[\eta_0^r(2^{[\alpha k]}(t - t_k)) \cdot u_r] \right\|_{\ell_n^2 L_\tau^2}, \\ \mathbf{II}_k &\sim \varphi_1(r) 2^{2k} \sup_{t_k \in \mathbb{R}} \sum_{j=0}^{2k+10} \left\| \frac{\eta_j(\tau - n^2)}{|\tau - n^2 + i2^{[\alpha k]}|} \mathbf{1}_{I_k}(n) \mathcal{F}[\eta_0^r(2^{[\alpha k]}(t - t_k)) \cdot u_r] \right\|_{\ell_n^2 L_\tau^2}, \\ \mathbf{III}_k &\sim \varphi_2(r) 2^{4k} \sup_{t_k \in \mathbb{R}} \sum_{j=0}^{2k+10} \left\| \frac{\eta_j(\tau - n^2)}{|\tau - n^2 + i2^{[\alpha k]}|} \mathbf{1}_{I_k}(n) \mathcal{F}[\eta_0^r(2^{[\alpha k]}(t - t_k)) \cdot u_r] \right\|_{\ell_n^2 L_\tau^2}. \end{aligned}$$

Here,  $\varphi_1(r)$  and  $\varphi_2(r)$  are given by

$$\varphi_1(r) = r^{-\frac{1}{2}}|r-1|^{\frac{1}{2}} \quad \text{and} \quad \varphi_2(r) = r^{-\frac{1}{2}}|r-1|^{\frac{3}{2}}$$

such that  $\lim_{r \rightarrow 1} \varphi_i(r) = 0$  for  $i = 1, 2$ .

It remains to deal with  $\eta_0^r$  in the integrand. Let  $\gamma : \mathbb{R} \rightarrow [0, 1]$  be a smooth cutoff function supported on  $[-1, 1]$  such that  $\gamma(t) \equiv 1$  on  $[-\frac{1}{4}, \frac{1}{4}]$  and  $\sum_{m \in \mathbb{Z}} \gamma(t - m) \equiv 1$ . Note that we have  $\gamma(t) = \gamma(t) \cdot \eta_0(t)$ .

Fix  $t_k \in \mathbb{R}$  in the following. By Fundamental Theorem of Calculus, we have

$$\eta_0^r(2^{[\alpha k]}(t - t_k)) - \eta_0(2^{[\alpha k]}(t - t_k)) = \int_1^{r^{-1}} \zeta_s(2^{[\alpha k]}(t - t_k)) ds, \quad (8.13)$$

where  $\zeta_s(t) = s^{-1}\zeta(st)$  with  $\zeta(t) = t\eta_0'(t) \in \mathcal{S}(\mathbb{R}_t)$ . Note that the left-hand side of (8.13) is supported on  $\{t : t - t_k = O(2^{-[\alpha k]+2})\}$  for  $r$  close to 1. Then, denoting the right-hand

side of (8.13) by  $F(t - t_k)$ , we have

$$\begin{aligned} F(t - t_k) &= F(t - t_k) \sum_{|m| \leq C} \gamma(2^{[\alpha k]}(t - t_k) - m) \\ &= F(t - t_k) \sum_{|m| \leq C} \gamma(2^{[\alpha k]}(t - t_k) - m) \cdot \eta_0(2^{[\alpha k]}(t - t_k) - m). \end{aligned} \quad (8.14)$$

Then, by Minkowski's integral inequality and Lemma 3.1<sup>9</sup> twice (for  $\zeta$  after Minkowski's integral inequality and for  $\gamma$ ), it follows from (8.12), (8.13), and (8.14) that

$$\begin{aligned} \|\mathbf{P}_k D_{\frac{1}{r}}(u_r)\|_{N_k^\alpha} &\leq \tilde{\varphi}_0(r) \|\mathbf{P}_k u_r\|_{N_k^\alpha} + \tilde{\varphi}_1(r) 2^{4k} \|\mathbf{P}_k u_r\|_{N_k^\alpha} \\ &\leq \tilde{\varphi}_0(r) \|\mathbf{P}_k u_r\|_{N_k^\alpha} + \tilde{\varphi}_1(r) 2^{4K} \|\mathbf{P}_k u_r\|_{N_k^\alpha}, \end{aligned} \quad (8.15)$$

where  $\tilde{\varphi}_0$  and  $\tilde{\varphi}_1$  are continuous functions defined on a neighborhood of 1 such that

$$\lim_{r \rightarrow 1} \tilde{\varphi}_0(r) = 1 \quad \text{and} \quad \lim_{r \rightarrow 1} \tilde{\varphi}_1(r) = 0.$$

Here, we used our assumption that  $\mathbf{P}_{\leq K} u_r = u_r$ . Finally, by summing over  $k \in \mathbb{Z}_+ \cap [0, K]$ , we obtain (8.8).

Next, we prove (8.7). Given  $\tilde{\varepsilon} > 0$ , let  $\tilde{u}$  be an extension of  $u$  outside  $[-T_0, T_0]$  such that

$$\|\tilde{u}\|_{N^{s,\alpha}} \leq \|u\|_{N^{s,\alpha}(T_0)} + \tilde{\varepsilon}.$$

Note that we have  $\mathbf{P}_{\leq K} \tilde{u} = \tilde{u}$  as before. Then,  $D_r(\tilde{u})$  is an extension of  $D_r(u)$  outside  $[-rT_0, rT_0]$  and thus we have

$$\|D_r(u)\|_{N^{s,\alpha}(rT_0)} \leq \|D_r(\tilde{u})\|_{N^{s,\alpha}}.$$

Hence, it suffices to prove

$$\limsup_{r \rightarrow 1} \|D_r(\tilde{u})\|_{N^{s,\alpha}} \leq \|\tilde{u}\|_{N^{s,\alpha}}. \quad (8.16)$$

Noting that (8.15) holds for functions independent of  $r$  (in place of  $u_r$ ), we see that (8.16) follows from (8.15). This completes the proof of Lemma 8.1.  $\square$

From (8.1), (8.2), and (8.3), we have

$$X(T)^2 \leq C_1 \|u_0\|_{H^s}^2 + C_2 \left( T^\theta M^{c(s)} X(T)^4 + M^{-d(s)} X(T)^4 + T^\theta X(T)^6 \right)$$

for  $T \in (0, 1]$ . With  $R = C_1^{\frac{1}{2}} \|u_0\|_{H^s}$ , first choose  $M = M(R)$  sufficiently large such that

$$C_2 M^{-d(s)} (2R)^2 < \frac{1}{4}. \quad (8.17)$$

Then, choose  $T_0 = T_0(R) \leq 1$  sufficiently small such that

$$C_2 T_0^\theta (M^{c(s)} (2R)^2 + (2R)^4) < \frac{1}{4}. \quad (8.18)$$

In view of Lemma 8.1, a continuity argument yields  $X(T) \leq 2R$  for  $T \leq T_0$ . Hence, from (8.1), we obtain

$$\|u\|_{F^{s,\alpha}(T)} \lesssim \|u_0\|_{H^s} \quad (8.19)$$

for  $T \leq T_0(\|u_0\|_{H^s})$ .

We are now ready to prove existence of solutions to (1.3) for  $u_0 \in H^s(\mathbb{T})$ ,  $s > -\frac{1}{8}$ . Given  $u_0 \in H^s(\mathbb{T})$ ,  $s > -\frac{1}{8}$ , let  $u_{0,n} = \mathbf{P}_{\leq n} u_0$  for  $n \in \mathbb{N}$ . Then, we have  $u_{0,n} \in H^\infty(\mathbb{T})$  and  $u_{0,n} \rightarrow u_0$  in  $H^s(\mathbb{T})$ . Moreover, by the classical well-posedness theory, there exist smooth

<sup>9</sup> The implicit constant in Lemma 3.1 is independent of  $s \in [1, r^{-1}]$  (or  $[r^{-1}, 1]$ ) for  $r$  close to 1.

global solutions  $u_n \in C(\mathbb{R}; H^\infty(\mathbb{T}))$  to (1.3) with  $u_n|_{t=0} = u_{0,n}$ . Then, we have the following compactness lemma for these solutions  $\{u_n\}_{n \in \mathbb{N}}$ .

**Lemma 8.2.** *Given  $u_0 \in H^s(\mathbb{T})$  for some  $s > -\frac{1}{8}$ , let  $u_n$  be the smooth solutions to (1.3) with  $u_n|_{t=0} = u_{0,n}$  as above. Then, the set  $\{u_n\}_{n \in \mathbb{N}}$  is precompact in  $C_T H^s := C([-T, T]; H^s(\mathbb{T}))$  for  $T \leq T_0$ . Here,  $T_0 = T_0(\|u_0\|_{H^s}) > 0$  is given by (8.18).*

*Proof.* It follows from Lemma 3.3 and (8.19) that  $\{u_n\}_{n \in \mathbb{N}}$  is uniformly bounded in  $C([-T, T]; H^s(\mathbb{T}))$ . Given  $\varepsilon > 0$ , we claim that there exists  $N_0 \in \mathbb{N}$  such that

$$\|(I - \mathbf{P}_{\leq N_0})u_n\|_{C_T H^s} < \varepsilon \quad (8.20)$$

for all  $n \in \mathbb{N}$ . In order to prove (8.20), we need to use the smoothing effect of the energy estimates. Namely, there exists small  $\delta > 0$  such that the estimates (7.8), (7.20), and (7.26) hold even with less regularity of  $r = s - \delta < s$  on the right-hand sides. See Remark 7.7. Let  $M$  and  $T < T_0$  be as in (8.17) and (8.18). Then, from (the proofs of) Propositions 7.3, 7.5, and 7.6 with (8.19), we have

$$\begin{aligned} & \left| \|\mathbf{P}_{>N} u_n\|_{E^s(T)}^2 - \|\mathbf{P}_{>N} u_{0,n}\|_{H^s}^2 \right| \\ & \lesssim (T^\theta M^{c(s)} + M^{-d(s)}) \|\mathbf{P}_{>N} u_n\|_{F^{r,\alpha}(T)}^2 \|u_n\|_{F^{r,\alpha}(T)}^2 + T^\theta \|\mathbf{P}_{>N} u\|_{F^{r,\alpha}(T)}^2 \|u_n\|_{F^{r,\alpha}(T)}^4 \\ & \lesssim C(\|u_0\|_{H^s}) N^{-2\delta} \rightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$ , uniformly in  $n \in \mathbb{N}$ . Hence, we have

$$\|\mathbf{P}_{>N} u_n\|_{C_T H^s}^2 \leq \|\mathbf{P}_{>N} u_n\|_{E^s(T)}^2 \lesssim \|\mathbf{P}_{>N} u_0\|_{H^s}^2 + C(\|u_0\|_{H^s}) N^{-2\delta} \rightarrow 0 \quad (8.21)$$

as  $N \rightarrow \infty$ , uniformly in  $n \in \mathbb{N}$ . This proves (8.20).

Fix  $\varepsilon > 0$ . By (8.20), there exists  $N_0 > 0$  such that  $\|\mathbf{P}_{>N_0} u_n\|_{C_T H^s} < \frac{\varepsilon}{3}$  for all  $n \in \mathbb{N}$ . Clearly,  $\{\mathbf{P}_{\leq N_0} u_n(t)\}_{n \in \mathbb{N}}$  is precompact in  $H^s(\mathbb{T})$  for each  $t$ . Moreover, by Lemma 3.3, (4.3), Proposition 6.1, and (8.19), there exists  $\theta > 0$  such that

$$\begin{aligned} & \|\mathbf{P}_{\leq N_0} u_n(t + \delta) - \mathbf{P}_{\leq N_0} u_n(t)\|_{H^s} \\ & \leq \|(S(\delta) - 1)\mathbf{P}_{\leq N_0} u_n(t)\|_{H^s} + \left\| \int_t^{t+\delta} S(t + \delta - t') \mathbf{P}_{\leq N_0} \mathfrak{N}(u_n)(t') dt' \right\|_{H^s} \\ & \lesssim \delta N_0^2 \|u_n\|_{C_T H^s} + \delta^\theta \|u_0\|_{H^s}^3 \leq C(N_0, \|u_0\|_{H^s}) \delta^\theta \end{aligned}$$

for all  $n \in \mathbb{N}$ . Namely,  $\{\mathbf{P}_{\leq N_0} u_n(t)\}_{n \in \mathbb{N}}$  is equicontinuous with values in  $H^s$ . By Ascoli-Arzelà compactness theorem,  $\{\mathbf{P}_{\leq N_0} u_n\}_{n \in \mathbb{N}}$  is precompact in  $C([-T, T]; H^s(\mathbb{T}))$ . Hence, there exists a finite cover by balls of radius  $\frac{\varepsilon}{3}$  (in  $C_T H^s$ ) centered at  $\{\mathbf{P}_{\leq N_0} u_{n_k}\}_{k=1}^K$ . Then, the balls of radius  $\varepsilon$  (in  $C_T H^s$ ) centered at  $\{u_{n_k}\}_{k=1}^K$  covers  $\{u_n\}_{n \in \mathbb{N}}$ .  $\square$

In view of Lemma 8.2, we can extract a subsequence, which we still denote by  $\{u_n\}_{n \in \mathbb{N}}$ , converging to some  $u$  in  $C([-T, T]; H^s(\mathbb{T}))$ . On the one hand, in view of the uniform tail estimate (8.21), this subsequence  $\{u_n\}_{n \in \mathbb{N}}$  also converges in  $E^s(T)$ . On the other hand, by possibly making  $T$  smaller, (8.1) and (8.2) with (8.19) yield

$$\|u_n - u_m\|_{F^{s,\alpha}(T)} \lesssim \|u_n - u_m\|_{E^s(T)}.$$

Hence,  $\{u_n\}$  converges to  $u$  in  $F^{s,\alpha}(T)$ . Finally, by applying Proposition 6.1 to

$$\begin{aligned} \mathfrak{N}(u_n) - \mathfrak{N}(u) &= \mathfrak{N}(u_n, u_n, u_n) - \mathfrak{N}(u, u, u) \\ &= \mathfrak{N}(u_n - u, u_n, u_n) + \mathfrak{N}(u, u_n - u, u_n) + \mathfrak{N}(u, u, u_n - u), \end{aligned}$$

we see that  $\{\mathfrak{N}(u_n)\}$  converges to  $\mathfrak{N}(u)$  in  $N^{s,\alpha}(T)$ . Hence, the limit  $u$  satisfies (1.3) as a distribution. This completes the proof of Theorem 1.2.

9. NON-EXISTENCE OF WEAK SOLUTIONS TO THE CUBIC NLS BELOW  $L^2$ 

In this section, we present the proof of Theorem 1.1. We prove this by contradiction. Fix  $s \in (-\frac{1}{8}, 0)$  and  $u_0 \in H^s(\mathbb{T}) \setminus L^2(\mathbb{T})$ . Suppose that there exist  $T > 0$  and a solution  $u \in C([-T, T]; H^s(\mathbb{T}))$  to the standard cubic NLS (1.1) such that

- (i)  $u|_{t=0} = u_0$
- (ii) There exist smooth global solutions  $\{u_n\}_{n \in \mathbb{N}}$  to (1.1) such that  $u_n \rightarrow u$  in  $C([-T, T]; H^s(\mathbb{T}))$  as  $n \rightarrow \infty$ .

The main idea is to use the a priori bound to the solutions to the Wick ordered NLS (1.3) from Section 8 and exploit the fast oscillation introduced in the transformation (9.1) below.

By setting

$$v_n(t) = e^{-2it \int_{\mathbb{T}} |u_{0,n}|^2 dx} u_n(t), \quad (9.1)$$

we see that  $v_n$  is a solution to the Wick ordered NLS (1.3). Moreover, we have  $v_n|_{t=0} = u_n(0) \rightarrow u_0$  in  $H^s(\mathbb{T})$ . By the a priori estimate (8.19), and a slight modification of the proof of Lemma 8.2, there exists a subsequence  $\{v_{n_k}\}$  converging to some  $v$  in  $C([-T, T]; H^s)$ , where  $T = T(\|u_0\|_{H^s})$ .

Indeed, by setting  $R = \|u_0\|_{H^s} + 1$ , we obtain the a priori estimate (8.19):

$$\|v_n\|_{F^{s,\alpha}(T)} \lesssim R$$

for  $n \geq N_1$ . As for the proof of Lemma 8.2, we need to modify (8.20). In particular, we claim that, given  $\varepsilon > 0$ , there exist  $N_0, \bar{N} \in \mathbb{N}$  such that

$$\|(I - \mathbf{P}_{\leq N_0})v_n\|_{C_T H^s} < \varepsilon \quad (9.2)$$

for all  $n \geq \bar{N}$ . Indeed, we have the following in place of (8.21); given  $\varepsilon > 0$ , there exist  $N_0, \bar{N} \in \mathbb{N}$  such that

$$\begin{aligned} \|\mathbf{P}_{>N} v_n\|_{C_T H^s}^2 &\lesssim \|\mathbf{P}_{>N} u_0\|_{H^s}^2 + (\|\mathbf{P}_{>N} u_n(0)\|_{H^s}^2 - \|\mathbf{P}_{>N} u_0\|_{H^s}^2) + C(\|u_0\|_{H^s}) N^{-2\delta} \\ &< \varepsilon, \end{aligned} \quad (9.3)$$

for  $N \geq N_0$  and  $n \geq \bar{N}$ . Then, (9.2) follows from (9.3). Then, we can repeat the second half of the proof of Lemma 8.2 on  $\{v_n\}_{n \geq \bar{N}}$  and find a finite  $\varepsilon$ -cover  $\{v_{n_k} : n_k \geq \bar{N}\}_{k=1}^K$  of  $\{v_n\}_{n \geq \bar{N}}$ . Finally,  $\{v_n\}_{n=1}^{\bar{N}-1} \cup \{v_{n_k} : n_k \geq \bar{N}\}_{k=1}^K$  forms a finite  $\varepsilon$ -cover of  $\{v_n\}_{n \in \mathbb{N}}$ . Therefore,  $\{v_n\}$  is precompact in  $C([-T, T]; H^s(\mathbb{T}))$  and there exists a subsequence  $\{v_{n_k}\}$  converging to some  $v$  in  $C([-T, T]; H^s(\mathbb{T}))$ . In particular, note that

$$v(0) = \lim_{k \rightarrow \infty} v_{n_k}(0) = \lim_{n \rightarrow \infty} u_n(0) = u_0 \quad \text{in } H^s(\mathbb{T}). \quad (9.4)$$

Now, we are ready to prove a contradiction. The main idea is to exploit the faster and faster oscillation in (9.1) as  $n \rightarrow \infty$ . Let  $\phi \in \mathcal{D}(\mathbb{T} \times [-T, T])$  be a test function. Then, we have  $\langle u_n(\cdot, t), \phi(\cdot, t) \rangle_{L_x^2} \rightarrow F(t) := \langle u(\cdot, t), \phi(\cdot, t) \rangle_{L_x^2}$ . Moreover, we have  $F(t) \in L^1(\mathbb{R})$  since it is continuous and supported on  $[-T, T]$ . Hence, by Riemann-Lebesgue Lemma, we have

$$\begin{aligned} \left| \iint v_n \phi dx dt \right| &= \left| \int e^{-2it \int_{\mathbb{T}} |u_{0,n}|^2 dx} \langle u_n(\cdot, t), \phi(\cdot, t) \rangle_{L_x^2} dt \right| \\ &\leq \left| \int e^{-2it \int_{\mathbb{T}} |u_{0,n}|^2 dx} \langle u(\cdot, t), \phi(\cdot, t) \rangle_{L_x^2} dt \right| \\ &\quad + \int |\langle u(\cdot, t) - u_n(\cdot, t), \phi(\cdot, t) \rangle_{L_x^2}| dt \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ . Note that the second term tends to 0 in view of the assumption (ii):  $\|u - u_n\|_{C_T H^s} \rightarrow 0$ .

On the one hand,  $v_n$  converges to 0 in  $\mathcal{D}'(\mathbb{T} \times [-T, T])$ . On the other hand, there exists a subsequence  $\{v_{n_k}\}$  converging to  $v$  in  $C([-T, T]; H^s(\mathbb{T}))$ . Hence, we conclude that  $v \equiv 0$ . In particular, from (9.4), we have  $u_0 = v(0) = 0$ . This is clearly a contradiction to  $u_0 \in H^s(\mathbb{T}) \setminus L^2(\mathbb{T})$ . This completes the proof of Theorem 1.1.

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